

Chapter 16

Reflections on Policy

What Should We Do, and Who Decides?

Nitsa Movshovitz-Hadar

Abstract Although mathematics is essential to mathematics-education, and mathematics-education is essential to mathematics, these claims do NOT imply that mathematics and mathematics-education are the same. Actually, they are gradually growing apart. This chapter summarizes the views of its authors on the relationship between the mathematics and mathematics-education communities with respect to policy issues believed to be important to both communities.

One argues that the professional object for mathematics teachers should be viewed as the teaching and learning of mathematics rather than mathematics in itself. Knowledge and experiences from mathematics as a discipline is necessary but not sufficient to form sustainable policy. Hence policy should benefit from being informed by mathematics-education research to a larger extent than currently.

Another view states that instructional policy is only as good as its translation to classroom practice. Without appropriate support for teachers to make the significant changes in classroom instruction being asked of them, curricular initiatives are bound to fail. Mathematicians and mathematics educators can and should collaborate to provide support to teachers in implementation of good mathematics teaching.

Yet another claim is that unlike mathematics, mathematics-education is an applied social science, and therefore research in it should be judged to a large extent, by the successful implementation of its outcome.

Last but not least is a view of mathematics and mathematics-education as two quite different areas of study, attributing many of the disputes that have arisen between mathematicians and mathematics educators with regard to what school mathematics should be, to these differences.

In conclusion, it seems necessary for the mathematics-education community and the mathematic community at large, to join forces and formulate a core of common

With contributions by

Jonas Emanuelsson, University of Gothenburg, Gothenburg, Sweden

David Fischman, California State University, San Bernardino, USA

Azriel Levy, Hebrew University, Jerusalem, Israel

Zalman Usiskin, Emeritus, The University of Chicago, Chicago, IL, USA

N. Movshovitz-Hadar (✉)

Dep. of Education in Science and Technology, Technion IIT, Haifa 32000, Israel

e-mail: nitsa@technion.ac.il

agreements, upon which decision makers can be held accountable. Indeed, a difficult task, however without it there seem to be no hope for progress in the desired commonly agreed goal to improve the outcome of mathematics-education.

Keywords Accountability · Classroom instruction · Curriculum · Curriculum change · Experimental school teaching · Mathematics-education · Mathematics-education research · Mathematics-education practice · Pedagogical content knowledge · Policy · Policy issues in mathematics-education · Relationship between mathematics and mathematics-education · Teacher knowledge · Teacher education

Mathematics and Mathematics-Education Policy—Searching for Common Ground

Nitsa Movshovitz-Hadar

About the Relationship Between Mathematics and Mathematics-Education

The American Mathematical Society defined mathematics *implicitly* by its Mathematics Subject Classification index (MSC 2010). Mathematics-education appears as subject no. 97. But mathematics-education is not just a formal part of mathematics. Mathematics-education has been playing a central role in the development of mathematics since antiquity, and clearly, there is no future to mathematics without mathematics-education. Hence, one cannot but remain amazed at the small, almost negligible amount of time and effort the majority of contemporary research mathematicians invest in mathematics-education per se.

Not surprisingly, but nevertheless somewhat paradoxically, the MSC index (ibid.) includes mathematics sub-entries under “area 97: Mathematics-education.” But Mathematics is not just ‘by definition’ at the heart of mathematics-education. Obviously, mathematics plays a central role in mathematics-education, and clearly, there is no future to mathematics-education without mathematics. Hence, one cannot but remain amazed at the small, almost negligible amount of time and effort the majority of contemporary research mathematics educators invest in mathematics per se.

Unlike the set theoretical implication $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$, although mathematics is essential to mathematics-education, and mathematics-education is essential to mathematics, these claims do *NOT* imply that mathematics and math education are the same. Actually, they are gradually growing apart!

A similar paradoxical relationship between education and mathematics-education can easily be formed. Is it because as yet we have not defined clearly what mathematics-education is all about? Or, at least its policy?

Let us examine the relationship through a few analogies, which should be considered ‘with a grain of salt’.

In a way—

- Mathematics-education (abbr. ME) is to Mathematics, as Sentential Logic (abbr. SL) is to Mathematics. Both are Meta-mathematics. SL is about the language of, and about the nature of deduction in Mathematics. ME is about the comprehension of, and about the nature of didactics in Mathematics.
- Mathematics-education is to Mathematics, as conducting a concert is to composing the music. Both ME and conducting a concert are not a free creation. Both are subject to human and real-life constraints.
- Mathematics-education is to Mathematics and behavioral sciences, as Architecture is to Mathematics and natural sciences. Their designs and constructs are interdisciplinary.
- Lastly, Mathematics-education research is to Mathematics-education practice, as Medical research is to Medical practice. Both of these research domains identify basic symptoms, and run empirical studies to develop and test innovative remedies for common cognitive/bodily diseases. Their results are implemented by practitioners for the benefit of their target populations.

Anyhow, mathematics and mathematics-education are definitely *NOT* mutually exclusive disciplines. They are inseparable, and in many ways are also complementary.

About Policy Issues in Mathematics Education

As commonly understood, policy refers to the collection of governing principles and plans instituted in order to operate some system, or a group of individuals, or sometime even one person.¹ Unlike a law, which prohibits or enforces certain behavior, some policies merely guide and help decision making about actions that are most likely to achieve a desired outcome (e.g. to increase the visibility of mathematics as a field of study²) or avoid some negative effect that has been noticed (e.g. declining enrollment in mathematics programs (see footnote 2)). A policy can be considered as a “Statement of Intent”, not necessarily including implementation procedures.³

How likely is it to reach a consensus about policy among mathematicians and mathematics-educators, beyond the obvious intent: ‘to improve the outcome of mathematics-education’? A major consensus-seeking process was carried out by NCTM in preparing its policy book on mathematics-education (2000): *Principles*

¹Note that Mogens Niss, in his Chap. 15 in this book, defines policy as something much more practical: decisions and actions, rather than the principles on which these decisions and actions are based.

²Example taken from The Joint Policy Board for Mathematics (JPBM). <http://www.mathaware.org/about.jpbm.html>. Accessed 13 June 2013.

³Note, once again that this is less in accord with Mogens Niss’ definition that appears in Chap. 15 of this book.

and *Standards for School Mathematics*. Regretfully, one policy principle was left out: a commitment to periodic review and continuous discussion. (This is not saying that NCTM has no intent to do it. Rather the opposite is implied by the fact that NCTM published in the past three related important documents⁴).

Clearly, mathematics itself is growing very rapidly and it permeates almost all walks of life. Which policies could/should be adopted in order to adequately prepare future mathematicians and scientists for the explosion of mathematics knowledge?

And not less important—What policies are appropriate for the education of the majority whose career may not be directly related to mathematics, but will be greatly affected by it?

Mathematics has become multifaceted to the extent that two mathematicians specializing in two different areas find it difficult to communicate. Who, then, can be approached for a comprehensive view of contemporary mathematics? Or, for an overview of its essence? Who, then, is there to be able to tell: What is important to teach at the pre-university level? And why is it important to teach it? Obviously, only after the policy is clear, can one deal with the issues of how to do it, and who can do it. How else can we relate to questions such as: Why and what for are we teaching the Pythagorean Theorem? Why are we not teaching at least 1 % of its proofs? Nor its generalization obtained by replacing the squares on each side by any similar polygons or semi circles? Nor do we clearly explicate that the cosine theorem is actually its extension to triangles that have no right angle?

And why don't we expose school students to the fact that 2 is the only power for which there is a solution (in fact infinitely many ones) to the Diophantine equation $a^n + b^n = c^n$, which makes the Pythagorean Theorem unique indeed?

How else can we justify why Euler's Polyhedron Theorem $V + F = E + 2$ is absent from most school curricula? And the infinitude of primes, or the theorem about 2-coloring of every map that is formed by intersecting circles?

Are these less accessible? Less important than the solution of quadratic equations? Or are they not sufficiently beautiful? Less powerful in terms of applications???

Setting criteria for sifting priorities cannot be an informed process without commonly agreed upon 'foundations' of mathematics-education, and without a periodical consensus-seeking discussion.

We witness long and hot debates about including this topic or omitting another one from the curriculum, employing this teaching strategy or avoiding another one, but they seem to lead nowhere, as there has not been enough investment in laying out the basic assumptions, the 'axiomatic system' of mathematics-education.

⁴The three prior publications by NCTM are:

- Curriculum and Evaluation Standards for School Mathematics (1989), which outlined what students should learn and how to measure the outcomes.
- Professional Standards for Teaching Mathematics (1991), which includes best practices for teaching mathematics.
- Assessment Standards for School Mathematics (1995), which focused on employing accurate assessment methods.

Isn't this the only area in the MSC2010 index (*ibid.*) that so far no attempt has been made to axiomatize? Nevertheless, there have been attempts at forming some mathematics-education statements as meta-mathematical theorems (e.g.: Every mathematics theorem is a boxful of surprises, Movshovitz-Hadar 1988, p. 39; 1993, p. 267).

It seems necessary for the Mathematics-education community and the mathematic community at large, to join forces and formulate a core of common agreements, upon which decision makers can be held accountable. Indeed, a difficult task, however without it there seem to be no hope for progress in the desired commonly agreed goal to improve the outcome of mathematics-education.

Let's not take too seriously the analogy between a discussion of mathematics-education policy and the idea of axiomatizing it as a mathematics sub-area. Instituting a policy is usually based upon accumulated experience, beliefs and research-based evidence. A consensus-seeking process takes compromise. We ought to be ready to stick to it for a while, but we also have to be prepared to reconsider it in view of recent developments—not only in mathematics itself, but also in the technology that becomes available, and in developments in other related areas that mathematics-education is leaning upon.

Let me conclude this part by two open questions:

- Q1. Narrowing the gap between school mathematics and contemporary mathematics—Is it a non-realizable dream, or can some curriculum-policy encourage it?
- Q2. Alongside the teaching and learning of mathematics, should mathematics serve also as a vehicle for human-values education? Or are these two 'orthogonal'?

On the Professional Object of Teachers

Jonas Emanuelsson

I have participated in educational research with a special interest in the teaching and learning in and about mathematics for a more than the past decade. During last years I have mainly done classroom studies of learning in an international context. I am the head of a department specialized in education rather than mathematics. This is an environment where education and training of becoming teachers, many of them mathematics teachers, heavily influences the daily life at the department. How to bridge the gap between mathematicians and mathematics educators in the context of teacher education is on the agenda almost every day.

One general point I want to emphasize here is that the professional object of mathematics teacher is better viewed as the *teaching and learning of mathematics* rather than viewed as mathematics in itself. In their teaching teachers should be oriented towards how learners respond to the mathematics taught instead of towards the mathematical content in itself. The content in an educator's mind while teaching should be mathematics as understood and handled by their students. Both working

teachers and pre-service teachers should, in my view, acquire sensitivity to discern the learners' understanding of mathematics. An integral part of policy in relation to mathematics education should consequently be directed towards the support of building knowledge in and about the learners' perspective on mathematics. Formulated in this vain policy have a possibility to become more helpful in answering questions on what teachers should know and hence what should be taught in teacher education programs.

At Gothenburg University prospective teachers study mathematics in one part of the organization (faculty of Science) and mathematics-education in another (faculty of Education). Chalmers University of Technology is the host for another teacher education program. At both universities, the mathematicians teaching mathematics in the teacher education programs and the mathematics educators teaching in the same programs do not always have constructive dialogues. There are elements of struggle for students and resources rather than a constructive collaboration aiming at catering for the best quality in teacher education. Many mathematicians tend to see the teaching of mathematics as something you learn in the course of teaching rather than something that can be informed by mathematics-education research or by extensive and documented experience from teaching in schools. On the other hand mathematics educators often see mathematicians as only interested in those students who are of "the right stuff" to become research students in mathematics and hence, in their view, tend to neglect a major part of the students.

Teacher education at the two universities of Gothenburg broadly follows three different paradigms or approaches with respect to teaching. These approaches are not mutually exclusive. Instead they usually exist side-by-side often within the same program and sometimes even in the same course. These paradigms can be summarized as follows:

- Teaching through knowing mathematics more solidly. The capability to teach improves with mere practice
- Learning to teach by imitating exemplary mathematics teaching
- Learning to teach by drawing upon research in mathematics-education

These paradigms place mathematics in very different roles. In the first paradigm mathematical knowledge is placed at the centre. Knowledge about teaching and the students learning falls to the background and is diminished to something that is personal and learned through practice. The second approach tends to underemphasize the systematic knowledge of both mathematics and mathematics-education research. If not balanced with solid knowledge in mathematics as a discipline the third approach runs the risk of being mathematically shallow. One could ask if this oversimplified description of different types of approaches illustrates the often argued gaps between mathematicians, mathematics-education researchers and mathematics educators?

There is considerable agreement that an effort is needed to bridge the gap between mathematicians, mathematics educators and researchers in mathematics-education as to how the nature of mathematics and mathematics-education are perceived. The bringing of people together in a joint knowledge building process is

foolish to argue against, and we should strive to bring it about in our respective communities. However I would like the different communities scrutinize what we possibly can gain from such bridging and how we should proceed to accomplish that. These are important and I think fair questions.

Mathematics-education, in my understanding, is best viewed as a hybrid science with a strong foundation in social science but with connections to mathematics both as a discipline and a school topic. The objects of study are human behavior, reasoning, problem solving while keeping a sharp focus on mathematical content as perceived, handled and treated by humans and organizations. We use methods and theories, sometimes fine-tuned to fit our specific purposes but borrowed from other social sciences rather than from mathematics in our efforts in understanding education, teaching, instruction, learning in and about mathematics. Our objects of research hence come from the human, cultural, social and psychological world. They do not belong to the world of mathematics. Mathematics-education is hence hard to conceptualize as an applied form of mathematics. Deep knowledge of mathematics is of course a necessary but not sufficient condition to practice mathematics-education research (or to teach it).

When forming policy about educational issues such as curriculum development, teacher education (both content and form), I argue that policy should be informed by research and practice in mathematics-education to a larger extent than presently.

To sum up I want to pose a series of questions that address issues raised above and in the paper by Mogens Niss in Chap. 15 this volume (the enumeration of open questions is continual throughout this chapter).

- Q3. In the discussion on policy e.g. teacher education policy we often state that results and experiences from both research in mathematics and research in mathematics-education are much needed. Furthermore we often argue that policy can benefit from bridging the gap between these two fields. What more precisely can we expect, or hope to, gain from such bridging?
- Q4. On what basis can we make well informed decisions on what to include in school curriculum and curriculum for (mathematics) teacher education? At the lowest level and at a minimum I argue that we would like policy makers, curriculum developers, teachers and becoming teachers to know something stable and systematic about:
- (a) Different ways of teaching mathematics and the corresponding learning that might be occurring (in relation to different aspects of mathematics, in different settings, during different contextualization, with kids of different backgrounds and experiences and so on).
 - (b) The targeted age groups ways of using mathematics in everyday situations (also outside schools and other institutions).
 - (c) Adults ways of using mathematics in future life (everyday life, in further academic studies and in different professions).
 - (d) Ways of using mathematics in other school subjects.
 - (e) Ways of using mathematics in other scientific disciplines.

- (f) We also need to know mathematics in terms of its historical development, use, nature, structure, affordances and constrains, as well as facts and procedures.

I am confident that mathematicians can contribute in answering these questions and that discussion on a shared arena can improve the answers.

I want to know, hear, and see more contributions from mathematicians. The idea and effort of building relations between education (as a practice and as a university discipline) and the mathematics discipline gave rise to mathematics-education. Research in mathematics-education is now developing as a discipline on its own with its own journals, conferences and organizations. I want to defend this as a specific area of expertise and invite mathematicians to take part. As other authors in this section argue I agree that mathematicians should become more active in the field of mathematics-education and participate in conferences and publish in mathematics-education research journals.

From Policy to Practice

David Fischman

Instructional policy is only as good as is its translation to classroom practice; it is useless to have stellar curriculum and instructional approaches determined at the national level, which then go through multiple interpretations and simplifications until they reach the classrooms of most teachers as a set of sterile packages of information and rules, which are then implemented as a laundry list of skills and algorithms. A typical chain of interpretations in the United States involves federal policy makers (whose policies determine allocation of funds to states as well as instructional standards), state policy makers (who refine instructional standards and determine state assessments of students), district mandates, school imperatives, and finally—finally!—the teacher, who puts all of this into practice in the classroom. In every consideration of mathematics and mathematics education policy, it is imperative that we consider the practical consequences for teachers and their students. Ongoing policies with an emphasis on multiple choice tests, along with the common view of school mathematics as computational, have continued to support teaching that is focused almost exclusively on computation and students who are afraid of mathematics, or if not, they love it because school mathematics primarily (in their experience) is algorithmic and computational, with few opportunities for creative thinking and little demand for real understanding.

On a recent plane trip, my neighbor asked what I do. When I responded: “I teach mathematics”, he replied: “Oh, I’m *crazy* about math!” He went on to tell me that he liked to play with numbers and find out all kinds of things about them. While his level of knowledge of mathematics might not have been very advanced, his teacher(s) had seemingly succeeded in the goal of “. . . lead[ing] students to appreciate the power and beauty of mathematical thought” (Dreyfus and Eisenberg 1986).

Some of us (mathematicians and mathematics educators) love mathematics for its power to help us understand and tame the world around us, others for the beauty of the structures of mathematics. Is it not incumbent upon us to support the creation and implementation of policies and curricula that lead the majority of students to view mathematics as powerful and beautiful?

Unfortunately, these goals are often at odds with the perceived need for accountability in the education system, which generally translates to measuring teachers' effectiveness by their students' scores on standardized, state-sponsored, tests of skill in applying mathematical algorithms and accurate computation. This behaviorist approach to education was lamented by Ted many years ago (Eisenberg 1975), and its use has only increased since then. He notes: "Behaviorists claim that education is an observable change in behavior, which is measurable, and hopefully permanent. . . . The student must be able to 'do something' as a result of instruction." And he goes on to say "It is incredulous that the State Departments of Instruction confuse education with training." Where is the "beauty and power" of mathematics in this approach? It is lost in the avalanche of paperwork, skills testing, and fear generated by teacher and administrator evaluation based almost exclusively on student scores in such tests.

With the advent of the Common Core State Standards, there is hope that student assessment in the United States will shift its focus to assessing understanding of mathematics, the ability to synthesize and apply mathematical ideas, and the ability to engage in and articulate mathematical thinking. How will school districts, teachers, and university mathematics educators respond to these changes and to the challenges that come along with them? It seems self-evident that in order to teach mathematics well, a teacher must have a deep understanding and appreciation of mathematics; mathematicians as well as mathematics educators will not, in general, argue with this statement. However, this statement is unfortunately vague—what constitutes "deep understanding"? Is such understanding developed by an undergraduate degree in mathematics? An advanced degree in mathematics? Special types of courses in mathematics education? A great deal of work has been done to answer these questions and to define appropriate concepts (for example, a synthesis of work involving pedagogical content knowledge can be found in Graeber and Tirosh 2008), but while there has been progress in understanding what is an appropriate knowledge base for a good mathematics teacher, there is still no consensus nor a generally accepted definition of the required concepts. If we are to arrive at policies that generally lead to high quality teaching, we would do well to include both mathematicians and mathematics educators in these discussions; this seems to be an excellent area for productive collaboration of mathematicians and mathematics educators to enrich the field in partnership.

Currently, mathematicians seldom view themselves—or are viewed by mathematics educators—as able to contribute to these discussions. Pre-service teachers typically are taught mathematics content by mathematicians, and teaching methods by mathematics educators; graduate programs all too often continue this separation of areas. Teachers, having been educated in a culture that separates content from pedagogy, are sent out into the professional world inadequately prepared to merge the two into a productive instructional program.

Thus teachers are asked to teach content for which they have been insufficiently prepared and students continue to learn mathematics primarily as a collection of procedures rather than a vital, deep, and beautiful discipline, and many very quickly develop a poor attitude toward what they believe to be mathematics.

At the other end of the spectrum are the mandated content and practice standards for students. In the US, state content standards are frequently far higher than the mathematics in actual classroom instruction, although not necessarily coherent or designed to lead to a good understanding of the fundamentals of mathematics. In recent years, there has been a well-orchestrated effort to construct common core content standards that are mathematically coherent, and standards for mathematical practice as a framework to support high quality mathematics work in the classroom. In addition to the standards document itself, the writing team is involved with writing supporting documents to clarify the content, and in designing assessments that reflect this approach and content.

What support will be provided to teachers as they are asked to change their approach to mathematics and to teaching mathematics, and to improve their content knowledge in order to bring their instruction to the level expected by the new standards? Over the years, various reforms have been designed and introduced with great fanfare and high hopes—only to crash on the rocks of classroom reality.

Teachers are the bridge between policy and practice. Ultimately, it is classroom instruction that makes or breaks a student's education—but classroom practice cannot be legislated. There is a human process that must take place in order to enhance instruction, and policy makers must consider this process at least as important as creating standards documents and determining textbooks. Without adequate support for teachers to make this transition, and time to practice new approaches without fear of immediate criticism, nothing substantive can change in classrooms throughout the country.

What is the role of mathematicians in this story? Typically mathematicians engage in the doing of mathematics, and leave the education to those who prefer to focus on education. As a research mathematician, that was certainly my approach to mathematics education for years. But then—if we mathematicians are unwilling to contribute to the world of education, what right have we to complain that it is done poorly? If we wish to see K-12 students learn mathematics as we believe it should be learned, it is up to us to participate in the design and implementation of good curricula and support what we understand to be good mathematical practice. In order to do this effectively, we need to learn more, and become more reflective, about mathematics education.

Clearly, research in mathematics and research in mathematics education are very different animals. Having “grown up” professionally as a mathematician, and currently learning to engage in high-quality mathematics education research, I have experienced the enormous differences in these two types of research; some of these differences are described in other papers in this monograph (e.g. other contributions to this policy discussion).

And yet. . . both revolve around mathematics and the doing of mathematics, albeit from different perspectives and in different contexts. Without a serious exchange of ideas between mathematicians and mathematics educators, both disciplines are less than they might be. There are mathematics faculty who have two distinct standards for mathematics: they themselves carry out first-rate mathematics research, but for the majority of their students they hold little hope of a deep understanding of mathematics, and the kind of teaching in which they engage does not lead to students stretching their thinking and developing good mathematical practice. We also know mathematics educators whose teaching revolves to a great extent around process and pedagogical strategies at the expense of content. Would we not be better off if there were more education in the world of the mathematician, and more mathematics in the world of the educator?

Some examples of university policies that support such collaboration include funding travel of mathematics faculty to meetings and conferences focusing on teaching, support for education and content faculty to collaborate on teaching, research, and grant activities, mentoring of junior mathematics faculty by senior mathematics and education faculty in learning about education and grant opportunities, and last but not least: recognition in the promotion and tenure process for mathematics faculty who engage in research and professional development activities related to mathematics education. In order that involvement in the work of education not be relegated only to veteran, tenured, mathematicians or to adjunct faculty or lecturers who teach mathematics education courses because they have no choice, university policy would need to change in many universities to include at least some of the items listed here. In order for young tenure-track mathematicians to become involved in education, their departments/colleges/universities will need to create policies (particularly regarding promotion and tenure) that support their participation in such activities. This would in turn require a major change in perspective, and would be potentially quite contentious—but without this sort of change, there is little hope that large numbers of mathematicians will become involved in education.

Some questions for consideration (the enumeration of open questions is continual throughout this chapter):

- Q5. What makes “good” mathematics education research?
- Q6. Would greater emphasis on mathematics as a discipline in policy decision-making lead to better, more effective, education policies?
- Q7. How can the education system reconcile its perceived need for accountability with its stated goal of teaching students to think mathematically and appreciate the power and beauty of mathematics? Can accountability systems be used towards this end?
- Q8. In what ways and to what extent should mathematicians be involved in mathematics education, and what policies should universities enact to provide recognition and encouragement to young mathematicians who participate in this work?

On the (Almost) Separate Roles of Mathematicians and Mathematics Educators

Azriel Levy

When one looks at the areas of mathematics and mathematics-education, almost everything mathematicians do has nothing to do with Math Education. And almost everything mathematics educators do has very little to do with what mathematicians do best, which is discover new mathematics. We can rely on mathematicians to tell us what parts of mathematics are more suitable for youngsters which intend to have a career in mathematics or science, but they have no advantage over mathematics educators when it comes to decide what mathematics to teach to youngsters who intend to become cooks or taxi drivers. Therefore many mathematicians are involved in mathematics-education, mostly in advanced high school education and the involvement goes down as you go down the ladders of the depth of the mathematics and of the school years. Of course, a mathematician can contribute much even to nursery school mathematics but this is a result of his personal qualities not of his expertise as a mathematician. The reasons for the higher involvement of mathematicians in advanced level high school mathematics are not only because this is the part of school mathematics which is closest to the mathematics done at the university but also because the graduates of high school mathematics are the beginning students of the university mathematics.

In Israel I did not evidence any rift or disputes between mathematicians and math educators because of the tacit agreement that the contents of the advanced level high school mathematics is determined mostly by the mathematicians, and as you go down the ladder of depth and school years the weight shifts more to the mathematics educators. Still, in the past some disputes occurred because the mathematicians and the mathematics educators belong to separate social groups, and a fad which carries over one group does not necessarily carry over the other. This was the case with the Cuisenaire rods.

A practical advantage which mathematics educators have over the mathematicians is that they control directly experimental school teaching, mostly by means of their graduate students, and as a result they can come up with realistic changes to the curriculum. Now we come to a problematic point, where the difference between mathematics and mathematics education comes to play. When a mathematician proves a good new theorem he writes up the proof and publishes it, and this is the end of the story of that theorem, and this is his contribution to mankind. Personally, this publication can mean to him a Ph.D. degree or a promotion at the university. When a mathematics educator carries out some experimental teaching and it turns out successfully he also writes up the description of what he did, publishes it, and gets promoted or gets his Ph.D. degree. The difference between the mathematician and the mathematics educator is that the theorem that the mathematician proved becomes a part of human knowledge and stays there forever, but the results of the experimental teaching are usually valid only for the present time and place,

and are not a lasting contribution to mankind.⁵ Therefore, the mere publication of the mathematics educator's innovation should not be the end of the story, and the mathematics educators should invest a part of their time and energy to make what they create, part of the current curriculum, or the prevalent teaching method. This is not always a very enjoyable activity in an educational system which is, by its very nature, conservative, but since mathematics-education is an applied social science its practitioners should not avoid that part of their work.

It is therefore up to the universities to judge the mathematics educators not only on their published work but also on the implementation of their results. This is not a problem of mathematics-education only, but of many applied sciences at the university. I understand that one can be a great surgeon but one will not be promoted without writing some mediocre papers. My feeling is that in Israel many good ideas in the area of mathematics-education ended with a Ph.D. thesis or a university promotion.

The same conflict between theorizing and applying is also evident in mathematics-education conferences. I attended several conferences on technology in mathematics-education. The main problem now is not the invention of new software and hardware but the mass implementation and the efficient use of those which became well known. Yet most of the talks were about new software and hardware or about novel uses of existing software rather than the efficient wide scale use of the bread and butter software. It is always more enjoyable to expose your bright new ideas than to describe the uphill fight of getting more and more students to use efficiently the available resources.

Mathematics and Mathematics Education: Two Quite Different Perspectives on the Same Subject

Zalman Usiskin

Allow me to begin with a comment on Mogens Niss view expressed in Chap. 15 of this volume. Unlike Mogens Niss, I distinguish a *policy* from a *practice*. A policy is something that is written down and is usually decided by a committee. Because a committee is involved, it is difficult to change a policy, whereas an individual can change a practice. Also, because it is decided by a committee, there are people from a variety of opinions on the committee; otherwise there is no need for a committee. Because of that variety, there are bound to be disagreements on the setting of policies, the beliefs of the committee members come into play, and beliefs trump data.

As an example of disagreement, many people in the United States think that students are worse today than students a generation or two ago even though widely

⁵Nevertheless, the reader may note the series of short publications by the Education Committee of the European Mathematical Society (EMS) on solid findings in mathematics education; One article of this series has been published in every Newsletters of the EMS since September 2011.

available data—from the National Assessment of Educational Progress—indicate that students in the United States—particularly elementary school students—are performing better than they ever have, an estimated one to two years ahead of where they used to be. Students are also taking more mathematics than ever before, so much calculus in high school that more students in the US take first-year calculus in high school than in college, and the best students fulfill their college mathematics requirements in high school.

So why do people not believe the data? I think it is because, in the past, college mathematics departments recruited their mathematics majors from their best calculus students, but today many of these students are never seen by those departments. In their place are students who in prior years would not have gone to college or, if they had gone, would not have had to take as much mathematics. So it is the case that the students college faculty see, are not as good as they used to be, but not because students in general are worse now than before.

When mathematics in upper secondary school was taken only by those who were going on to study in the physical sciences or engineering, there was not much dispute between mathematicians and mathematics educators because these students would all take calculus and so there was obvious preparation in school for that. It is with the notion that *all* students should study mathematics through secondary school that tensions have come to the forefront. The battle is of a common type in society when decisions have to be made that affect different groups: it is a turf war.

Everyone in Israel knows how bitter turf wars can be and how difficult they are to settle. But they must ultimately be settled and the directions for settling them are rather well-known even if difficult to establish. That is what we are trying to do here: discussions of our common goals and searching for common ground; frank discussions of our differences in a civil manner and in a way that clarifies the reasons for these differences and tries to erase misconceptions; all of this aiming at mutual respect and tolerance.

Our fields are not alike. Even a cursory look at journals in mathematics and mathematics-education shows these fields to be fundamentally different. For the most part, the ultimate objects of mathematics are concepts and problems; the ultimate objects of mathematics-education are students. The objects of mathematics are inanimate and eternal; the objects of mathematics-education are animate—indeed, often quite animated—and constantly changing. Truth in mathematics is established by a logical proof, while truth in mathematics-education is dependent on data, and data fluctuate, so a result in mathematics-education in one place might not apply to another.

Póya, an acknowledged expert in both fields, described the first step in solving a difficult problem as understanding the problem. The underlying problem that fuels mathematics-education and brings mathematicians into mathematics-education is the perception that students do not know as much mathematics as we would like them to know. There has never been a time or a place where this problem is not perceived. Even in Singapore and Shanghai people have these beliefs. Then the question is: Who can best address this problem?

Ted and Michael Fried pointed out in their paper (Eisenberg and Fried 2009) that mathematics educators do not agree on many fundamental questions, such as the

best ways to do research and what are important questions to ask, which is true, but the reality is even more frustrating because even on those things in which there might be agreement today, conditions could change tomorrow. There are disagreements in mathematics, too, such as whether computer proofs should be considered as valid or, if we want to go back into history, whether we want to allow the axiom of choice or an equivalent, but there is more agreement in mathematics than in mathematics-education, as one would expect of a natural science over a social science.

Unlike Jonas Emanuelsson and some others represented in the book, I find it very appealing to view mathematics-education as one type of applied mathematics. And like other fields that apply mathematics, such as statistics, computer science, physics, or operations research, although the field is grounded in mathematics, the problems that fuel mathematics-education emerge from the world outside of mathematics. Additionally, there are aspects to the field of mathematics-education that are not mathematical at all, and other aspects that are border-line.

The policy-maker dealing with mathematics curriculum, the area of my major work, deals with the selection of content to be covered in school, who should encounter that content, in what sequence, and at what age. Concerning the selection of content, is statistics mathematics? Is formal logic a part of mathematics? Is physics mathematics? In general, when if ever does applied mathematics cease to be mathematics? Should telling time be a part of the mathematics curriculum? What about reading tables of data or locating one's home town on a map? What about doing a logic puzzle such as a Sudoku puzzle? What about a discussion of lucky numbers and favorite numbers and unlucky numbers? Is computing using a calculator *doing* mathematics or *avoiding* it? Is conjecturing *mathematics* or is it *proto-mathematics*, that is, not the real thing but leading up to the real thing. These questions bring out differences both between and within our groups in what we think mathematics is, and differences in what we think is *real* or *good* mathematics.

Pólya's second step in problem solving he called "devising a plan", and one bit of his advice in this regard is to look at a related problem. A discussion like this one could involve statisticians and mathematicians rather than mathematics educators and mathematicians. We all know that while in some universities statisticians reside in a department of mathematics, in other universities they have their own department. The statisticians on my campus are all very knowledgeable about mathematics, but they do not view their discipline as a sub-branch of mathematics. My own background reflects the difference—I minored in statistics as an undergraduate in a department of mathematics and at no time in any of my statistics courses was I ever asked to examine a data set. Our distributions were all theoretical. In contrast, the statisticians on my campus feel that data is the starting point for all statistics, just as we in mathematics-education have to begin with the learner, the teacher, or the school situation, not with mathematics.

An effect of this difference in view is seen in the Common Core State Standards in the US, where statisticians had very little influence. Only a few years ago a committee of the American Statistical Association (ASA) published a report with detailed guidelines and examples for a curriculum in statistics K-12 (Franklin et al.

2007). The Common Core document references this report but it seems that is as far as the Common Core authors went. There is no statistics in grades K-5 and the statistics that is in grades 6-12 is not developed with anything near the care that is in the ASA document.

The difference in views of statistics reflects a fundamental difference between school mathematics and the mathematics as done even in the most inclusive college mathematics departments. School mathematics covers a far broader agenda than mathematics. Just as statistics involves such things as the design of experiments, a topic that is not on the radar screen of pure mathematics, the school mathematics teacher, and thus the school mathematics curriculum, is obligated to cover all the basics of the mathematical *sciences* that the public needs, including quantitative literacy and other topics that do not fit any sort of logical mathematical system. This might explain circle graphs or the metric system, both topics hard to fit into a discussion of the properties of numbers, are not mentioned in the recent Common Core standards in the United States and why much more attention is given to fractions, easily associated with division, than to decimals, a numeration system.

The natural sequence in mathematical research is logical. Accordingly, when I first wrote curriculum, I thought that all children would learn easily from a carefully-written well-explained logical mathematical argument. I was mistaken. The learner brings elements into learning mathematics that do not fit mathematical logic. Young children are not typically convinced by a logical argument. Even adults are not always convinced by logical arguments; they tend to view an argument as valid if they believe the conclusion, and invalid if they do not believe the conclusion. It's not too strange to feel that way. We have to teach students to have faith in mathematical logic and in mathematical systems; we have to teach students that one can proceed logically from a false statement to another false statement. We cannot assume that such thinking is innate.

The belief in the primality of a logical sequence in mathematics curriculum is related to what is meant by understanding of a mathematical concept. Does a person fully understand the division of fractions because they can derive the general rule from other properties? I would say they do not. The full understanding involves knowing alternate algorithms for finding the quotient, recognizing and being able to apply the division of fractions in problem situations, and being aware of the history of the idea. Some mathematics educators and psychologists might add to this the ability to represent division of fractions in some sort of iconic way. Mathematics educators tend to harbor a broader view of understanding than mathematicians.

Sticking to the notion that a logical sequence is needed in order to understand a mathematical concept can narrow what students encounter in their mathematics experience. There is no mention of infinite decimals in the Common Core, perhaps because some on the writing committee felt that an understanding of limits is needed to understand infinite decimals. Rarely does one find theorems such as the Isoperimetric Inequalities in the plane and 3-space before college even though they have many applications, probably because the proofs of these statements require college-level mathematics.

Agreeing that a logical sequence is not appropriate for all topics in school mathematics, some people hold the notion that the optimal sequence through elementary

mathematics should follow the sequence of their invention by mathematicians. How else can we explain that negative numbers do not appear in the Common Core curriculum until grade 6, after fractions and decimals? I remember well a conference in the early 1970s at Southern Illinois University where Frederique Papy spoke about how negative numbers were introduced in their elementary school mathematics curriculum in Belgium. In first grade, from the beginning of school, the teacher and students graph the high and low temperatures for each day. And around November the low temperatures first go below freezing, so voila (but not in Beer Sheva!)? Pure mathematics avoids everyday experiences, but mathematics-education not only cannot avoid them, it is well-advised to use them.

Although the fields of mathematics and mathematics-education are so different, in both fields we teach mathematics, and in both fields we think deeply about mathematical concepts, and we think that our perspective gives us special insights into that mathematics. Some of us—perhaps most of us here—would like to think that we straddle the fields, but for most people these differences constitute the way it is.

Thus we should not be surprised that there are conflicting views towards a host of curricular issues. In some places, the diversity of views is welcomed and there is mutual respect between mathematicians and mathematics educators. However, in the US, these conflicts have not been resolved; whoever is in power locally or nationally rules the day, and the conflicts are being played out as we speak in the implementation of the Common Core, where the stakes for agreement are higher than they have ever been. We should take advantage of the fact that mathematics is an international language and work together for the common good.

With the entry of mathematicians into the mathematics-education arena, one of the nicest things that has happened is that mathematicians have been outspoken in the view that elementary mathematical concepts can be quite complex, and that understanding these concepts is not trivial and requires deep thought.

But other mathematicians have not been so thoughtful. They know a little about mathematics-education but they think they know more than the people who have spent their lives in the field. Their writing is a combination of accurate statements and silly pronouncements, hidden behind mathematical arguments to exhibit their knowledge of theory.

It is also the case that there is nonsense research in mathematics-education. We need some of our mathematicians to speak out against those mathematicians who are preaching nonsense. And, at the same time, more of us mathematics educators need to speak out against nonsense in our field.

Reflective Summary

Nitsa Movshovitz-Hadar

The focus of this chapter, mathematics and mathematics-education policy, is multifaceted. The multitude of ideas and connections embedded in mathematics, and

the characteristics of research in mathematics as well as in mathematics-education make it almost impossible to cover many policy issues under the constraints of one chapter in a book or one panel in a symposium. We have touched upon several issues but more are left implicit, untouched or as open questions specified explicitly.

Nevertheless, mathematics teachers' preparation and the school curriculum are the two pillars of mathematics-education. Teacher education policy and curriculum policy are therefore at the heart of mathematics-education policy. Striving for vision may yield an overview of the essence of contemporary mathematics. In collaboration between mathematicians and mathematics-educators, such an overview may yield criteria to examine pre-university mathematics subjects for their educational potential. This would be an invaluable contribution towards a curriculum policy.

A collaborative process of investigation about the requisite mathematical knowledge of mathematics teachers, its acquisition and lifelong development (such as Gutfreund and Rosenberg 2012) will hopefully yield some new policy concerning teacher preparation.

To the extent that policy concerns curriculum and standards, it demands informed opinions on the kinds of fundamental issues we spoke about.

To the extent that it should make decisions about such things, it must face the difficulty of reconciling seemingly conflicting ends—for example, between being mathematically precise and rigorous and being intuitive or heuristic for the sake of creating steps towards further learning.

A forum where policy is discussed and decided becomes thus a natural occasion for discussing matters of common interest to mathematics-education as well as a context requiring mathematicians and mathematics educators to come together and bring with them their own special perspectives in order to sift priorities.

Pre-university mathematics-education is facing many challenges: e.g. creating motivation, and maintaining it; mathematics-education should go hand in hand with education for human values. Mathematics-education must go hand in hand with mathematics, namely exposing students of all ages to the broad spectrum of contemporary mathematics that permeates almost all walks of life, to the true nature of mathematics as an ever-growing body of knowledge, its applications, its beauty and its rich intellectual challenge.

Math education policy should be adopted to help in coping with these challenges. The “how to” will follow if the policy statements are clear.

References

- American Mathematical Society (2010). Mathematics Subject Classification index (MSC2010). <http://www.ams.org/mathscinet/msc/msc2010.html>. <http://msc2010.org>. Accessed 12 June 2013.
- ASSM (1995). *Assessment standards for school mathematics*. NCTM—National Council of Teachers of Mathematics, Reston, VA.
- CESSM (1989). *Curriculum and evaluation standards for school mathematics*. NCTM—National Council of Teachers of Mathematics, Reston, VA.

- Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the Learning of Mathematics*, 6(1), 2–10.
- Eisenberg, T. A. (1975). Behaviorism: the bane of school mathematics. *International Journal of Mathematical Education in Science and Technology*, 6, 163–171.
- Eisenberg, T., & Fried, M. N. (2009). Dialogue on mathematics education: two points of view on the state of the art. *ZDM. The International Journal on Mathematics Education*, 41, 143–149.
- Franklin, C., Kader, C., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) report: a pre-K-12 curriculum framework*. Alexandria: Am. Statist. Assoc.
- Graeber, A., & Tirosh, D. (2008). Pedagogical content knowledge: useful concept or elusive notion. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 1, pp. 117–132). Rotterdam: Sense Publishers.
- Gutfreund, H., & Rosenberg, Y. (Eds.) (2012). *Knowledge and training of secondary school mathematics teachers—a report of the committee on the knowledge-base for teaching mathematics*. Jerusalem: Israel Academy of Sciences and Humanities. <http://education.academy.ac.il/English/PublicationDetails.aspx?FromHomePage=true&PublicationID=27>. Accessed 12 June 2013.
- Movshovitz-Hadar, N. (1988). School mathematics theorems—an endless source of surprise. *For the Learning of Mathematics*, 8(3), 34–40.
- Movshovitz-Hadar, N. (1993). The false coin problem, mathematical induction and knowledge fragility. *The Journal of Mathematical Behavior*, 12, 253–268.
- PSSM (2000). *Principles and standards for school mathematics*. NCTM—National Council of Teachers of Mathematics, Reston, VA.
- PSTM (1991). *Professional standards for teaching mathematics*. NCTM—National Council of Teachers of Mathematics, Reston, VA.