

The 4-Weight Problem - A Pedagogical Account

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When you are unable to solve a problem, would you consider it foolish to attack its generalisation? Polya³ gives examples of problems which are easier to solve when generalised, and other examples have appeared in *Mathematics Teacher* (Alexander *et al.*¹, Walter and Brown⁶, Krulik⁷).

The "Weight Problem of Bachet de Meziriac" which follows is another such example. It is interesting in itself, and it has the advantage that both the problem and its solution are easily understood, even by students with little background in mathematics.

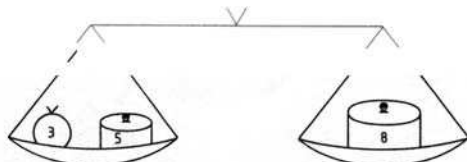
The Problem

A merchant had a 40-kg measuring weight that broke into four pieces as the result of a fall. When the pieces were subsequently weighed, it was found that the weight of each piece was a whole number of kilograms and that the four pieces could be used to weigh every integral weight between 1 and 40 kg. What were the weights of the pieces? (Dorrie²).

Note: Weights may be placed in either pan of the balance. For example, if an object placed in one pan together with a 5 kg weight, is balanced by an 8 kg weight, then the object weighs $8 - 5 = 3$ kg (see Fig. 1).

Common Procedures

When students are allowed free time to seek a solution, they



If load and 5 kg balance 8 kg, then load weighs 3 kg.

Fig. 1 Weighing a load of 3 kg on a balanced scale using 5 kg and 8 kg weights.

tend to focus on the numerical data — 4 weights and 40 kg. The procedures they commonly resort to include: wild guessing; writing down all quadruples of integers whose total equals 40, and attempting to find the correct quadruple by elimination; deriving that one of the weights must be 1 kg since otherwise it would be impossible to make up 39 kg; and similar ideas, most of which embody a related problem more specific than the original one, with the figures 4 and 40 "built in".

Some students reach the right solution by pure luck, but most students, sticking to the numerical data 4 and 40, find the problem frustrating. This is because the search for a systematic method which seems promising and both effective and not unreasonably long, seems to be endless. As we shall soon show, the disappointment at this point can easily be changed into a drive to look for fresh ideas.

(If you have not yet done so, try taking advantage of these hints to solve the problem.)

Generalisations of the Problem and Problems Analogous to the Original One

To overcome the frustration and develop a successful strategy, students should be made aware of the obstacle caused by sticking rigidly to the numerical data. They must be encouraged to depart from the given numbers and to reformulate the original problem. This exercise of reformulating the original problem and of stating analogous and more general ones can be, in itself, an extremely valuable activity.

The path to more general problems may involve the following questions:

- Is a solution unique? i.e. Are there more than one four-piece set which satisfy the conditions?
- Are there sets of more or less than four pieces, satisfying the conditions?
- Are there sets of any number of pieces which do not total 40 kg and satisfy the requirement of weighing every

integral weight between 1 and 40, between 1 and another number smaller or greater than 40?

Such questions lead to a reformulation of the original problem by breaking down its conditions into the following steps:

- Find sets of integral weights with which any integral load from 1 to 40 kg can be weighed.
- Find such sets with exactly four pieces.
- Find a four-piece set whose total equals 40 kg.

The reformulation does not assure the solvability of the problem, but it helps clarify it and illuminate it as a particular case of a more general problem:

For any natural number n , find at least one set of integral weights with which any integral load from 1 to n kg can be weighed; evaluate the different sets possible for each n (e.g. in terms of economy).

The general case, in turn, permits a systematic search of solutions for increasing values of n , starting from simple cases such as $n=2, 3$, etc.

This generalisation offers a strategy for attacking the problem in a way which seems more systematic than blind experimentation. Whether or not it will lead to a solution, as yet remains unknown. However, students should now feel more confident that they hold an end of the thread.

Solution of Particular Cases

We approach the general problem through systematic examination of particular cases.

The first non-trivial case is $n=4$. The two-piece set of weights $\{1, 3\}$ enables us to weigh any integral load between 1 and 4 kg (for a 2 kg load we place the load and the 1 kg weight on one pan and balance them with the 3 kg weight on the other pan; for a 4 kg load we place both weights on one pan). True, $\{1, 2, 3, 4\}$ is also a solution, but it is wasteful; it has more weights than needed. Any other solution like $\{1, 2, 3\}$ or $\{4, 6, 7\}$ is also less economical than $\{1, 3\}$.

For $n=5$ there is no two-piece set which fulfils the conditions of our problem, because with $\{2, 3\}$ 4 kg cannot be weighed, with $\{1, 4\}$ 2 kg cannot be weighed, and with two weights the total of which is less than 5, 5 kg cannot be weighed. If one or both weights are greater than 5, the requirement cannot be fulfilled either. Therefore, it follows that for $n=5$ at least three weights are needed. Moreover, by similar arguments, for any value of $n > 4$ at least three weights are needed. Now, is there a three-piece solution for $n > 4$?

It is natural to try solving for $n=5$ using the former findings for $n=4$. For students however, this is not at all spontaneous. They often tend to start each case anew and work towards an independent solution. Very rarely does it occur to them to take advantage of knowledge recently acquired. Let us, however, recall that for $n=4$, $\{1, 3\}$ is the solution.

What would be the most judicious addition to $\{1, 3\}$ so that the extended set would solve the case of $n=5$? Sets $\{1, 1, 3\}$ and $\{1, 2, 3\}$ meet the requirements, and the latter even covers the case of $n=6$.

What is the maximum value of n for a three-piece set? Would it be more expedient to add a 5 kg weight? We would then be able to weigh every integral load from 5 through 8 kg, but since $\{1, 3, 5\}$ provides two alternatives for a 4 kg load ($1+3$ and $5-1$) and for others too, one suspects it may be possible to do even better. The same is true if we add a 6, 7, or even an 8 kg weight to the set $\{1, 3\}$. By contrast, a 9 kg weight provides a unique solution for each integral load from 1 through 9 kg. This is because for loads of 1, 2, 3, 4 kg the subset $\{1, 3\}$ suffices, as we have seen above. For loads of 5, 6, 7, 8 kg, we subtract 4, 3, 2, 1 kg respectively from 9, where the composition of 1, 2, 3, 4 for the subtraction is again uniquely facilitated by the subset $\{1, 3\}$.

The new set $\{1, 3, 9\}$ actually provides a solution for cases up to $n=13$, because loads of 10, 11, 12 and 13 kg can be

weighed by adding to 9 either 1, 2, 3, or 4 kg respectively, assembled as before from the subset $\{1, 3\}$. The various combinations are illustrated in Figure 2. Clearly the set $\{1, 3, 9\}$ is the most economical set for $n=13$.

At this point students should be able to define an economical set for a particular value of n as a set which enables to weigh

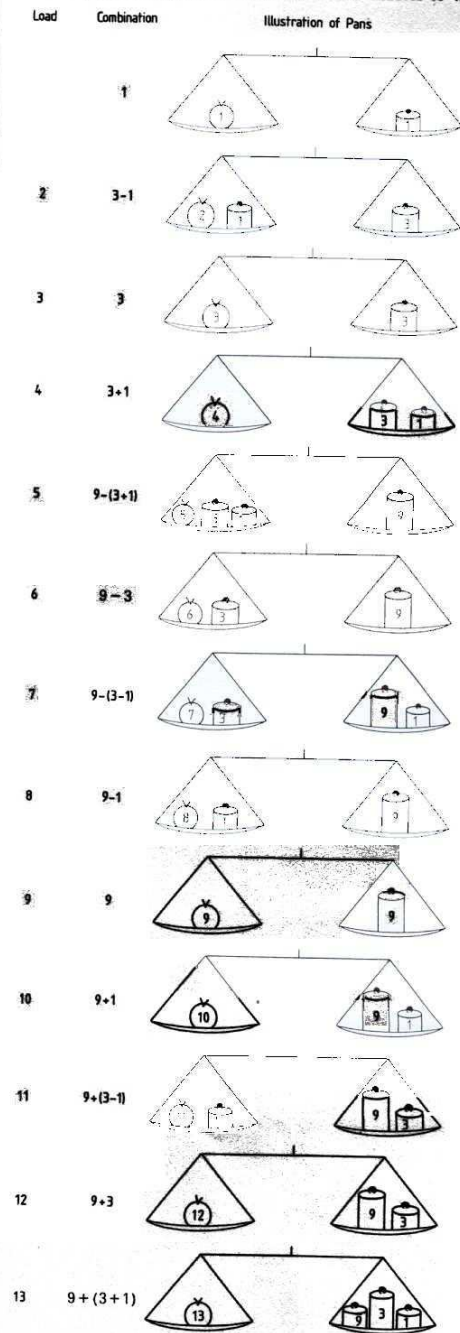


Fig. 2 Illustration of weighing 1-13 kg with 1, 3, 9 kg weights.

each load from 1 to n in one and only one way, and has the minimum number of pieces.

Now we can extend our reasoning to larger values of n . For $n > 13$ we need a set with at least four pieces. We can now proceed more rapidly by analogous considerations. To find the new four-piece set which is practicable for the maximum value of n , the additional piece x must permit weighing loads in the 1 to 13 kg interval in the same manner as the three-piece set, while loads of 14 kg and over would be weighed by subtracting 13 kg or less from x . Thus $x - 13$ must equal 14, so $x = 27$ kg. Because any number of kilograms between 1 and 13 can be assembled without the newly added 27 kg weight, with the new set we can uniquely weigh not only loads of 27 minus 1, 2, . . . , 13 kg, but also loads of 27 plus 1, 2, . . . , 13 kg. Namely the four-piece set {1, 3, 9, 27} provides a solution for $n = 40$, which happens to coincide with the value of n in the original problem, with which we started. How fortunate!

Solution of the General Problem

The arguments which lead to solutions for particular values of n are readily generalisable: Let the set $\{W_1, W_2, \dots, W_k\}$ be a finite set of integral weights by which any integral load g between 1 and a given upper limit G can be weighed in one and only one combination of some or all of the weights on the scales.

By adding a weight $W = 2G + 1$ to that set, the weighing interval is extended to $G + W = 3G + 1$, i.e. with the extended set $\{W_1, W_2, \dots, W_k, W\}$ any load g , where $1 \leq g \leq 3G + 1$ can be weighed in a unique way.

Proof

By the initial assumption any value of g such that $1 \leq g \leq G$ can be uniquely weighed by the subset $\{W_1, W_2, \dots, W_k\}$ of the extended set $\{W_1, W_2, \dots, W_k, W\}$. For values of g where $G + 1 \leq g \leq 2G + 1$, notice that the difference $d = W - g = 2G + 1 - g \leq 2G + 1 - (G + 1) = G$. Therefore the difference $d = W - g$

can be uniquely assembled by the original set. Now to get g we can subtract d from W . Finally, to reach any value of g where $2G + 2 \leq g \leq 3G + 1$ we have to add G kg at the most to the new weight of $W = 2G + 1$. Thus we have shown for any value of g which satisfies $1 \leq g \leq 3G + 1$ that the extended set provides a unique way of weighing.

This solution is a recursive one. Because we have already found a solution for $n = 40$, we can now apply the general solution to find a (non-wasteful) weighing-set for all n . For example, for $n = 41$, we extend the set of {1, 3, 9, 27} by adding to it $2 \times 40 + 1 = 81$. With the new set {1, 3, 9, 27, 81} we can weigh any integral load between 1 and 121 kg. In other words, this set solves the problem for $41 \leq n \leq 121$.

Summary

In this paper we proposed and demonstrated one way in which generalisation can be incorporated into high-school mathematics, that is generalisation as a mode of problem solving. In another paper (Hadar and Hadass³), we discuss a second way of incorporating generalisation into high-school mathematics, that is generalisation of a known topic as a trigger to the study of a new topic. A third way would be generalisation as a means of review. Challenging ideas for this alternative can be found in Alexander *et al.*¹

References

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