

# Sometimes Students' Errors Are Our Fault

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“The root of students’ frustration with mathematics is that they are so frequently wrong” (Wood 1982). We accept this thesis, along with the fact that taking a written test is not a relaxing experience. Therefore, as test designers, we must make an effort to minimize the amount of irrelevant obstacles students face when taking a test.

Several methods of constructing achievement tests in mathematics are well established (e.g., Merwin 1961; Suydam 1974), and teachers usually take them very seriously. However, the study of students’ errors can shed light on unforeseen obstacles that sometimes are unintentionally added to a test. In this article we share the lesson in editing that we learned from such a study. Classroom teachers might find the results interesting because so much of their evaluation of students is based on written tests, yet the average teacher cannot afford the time and effort involved in a careful analysis of students’ mistakes and may not even notice the kinds of editorial pitfalls we present.

The source of data was a sample of 300 examinees’ answer sheets to items on national mathematics tests administered to graduating high school students in Israel. Errors that occurred repeatedly were analyzed. We looked for a reasonable explanation of the errors’ underlying logic. Then we constructed a classification model with six categories (Movshovitz-Hadar et al. 1985a; 1985b). The errors were exceptionally puzzling. At first we thought them weird and extremely “illogical” until we looked back at the form of the test. There we recognized some editorial practices that offered a reasonable explanation of the kind we were after. To test the explanation,

we administered revised items in which these editorial pitfalls were eliminated. All the findings reported here are based on students’ better performance on the revised test items.

## Careless Printing

### Case 1

*In an isocoles triangle  $ABC$ ,  
 $AB = BC = 25$  cm;  $m\angle ABC = 38^\circ$ .  
 $\overline{AD}$  bisects  $\angle BAC$ . Find the radius  
of the circle circumscribing the triangle  
 $ADC$ .*

In figure 1, many students calculated the radius of the circle circumscribing triangle  $ABC$  instead of triangle  $ADC$ . In searching for the logic behind this mistake, we speculated that because in the question “ $ADC$ ” was typed on a separate line and because the sentence could reasonably end without it, students who were feeling test pressure and were eager to start working on the problem read it in haste and overlooked the word in the last line. Indeed, this mistake did not occur in a class that worked the same problem from a retyped paper in which “ $ADC$ ” was added to the previous line.

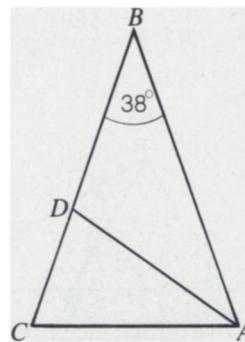


Fig. 1

Case 2

Given that  $\log x = 2\log 3 + \frac{1}{2}\log 16 - \log 27$  calculate the value of  $x$  without tables or calculators.

The following quotation from a student's workbook represents a very common mistake:

$$\log x = \log \frac{3^{21} \cdot 16^{1/2}}{27}$$

$$x = \frac{3^{21} \cdot 4}{27} = 3^{18} \cdot 4 = 27^6 \cdot 4 = \dots$$

The mistaken exponent of 3 seems to have only one explanation: The student read the first letter "l" of "log" as the numeral 1 and considered the expression "2log 3" as "21 log 3." It would be advisable to help students avoid such mistakes, for instance, by putting a multiplication dot between the coefficient and the word "log" (e.g., 2 · log 3).

**Misleading Figure**

Case 1

Triangle  $PQR$  is equilateral.  $A, B$  are two points on the continuation of  $\overline{RQ}$ .  $m\angle APB = 120^\circ$  [see fig. 2].

- Prove: (i)  $\angle APQ \cong \angle PBR$   
 (ii)  $\triangle APQ \sim \triangle PBR$

The following student's answer demonstrates a typical error leading to the desired conclusion:

$$m\angle PQR = m\angle PRQ = m\angle QPR = 60^\circ$$

and

$$m\angle APB = 120^\circ \text{ (given)}$$

implies

$$m\angle QPA = m\angle RPB = \frac{120^\circ - 60^\circ}{2} = 30^\circ.$$

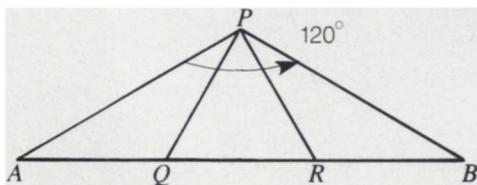


Fig. 2

We know

$$m\angle PQA = m\angle PRB = 120^\circ,$$

therefore

$$m\angle A = m\angle B = 30^\circ;$$

hence

$$m\angle APQ \cong \angle PBR \text{ Q.E.D. (i)}$$

and

$$\overline{PQ} \cong \overline{PR} \text{ (given).}$$

Therefore,

$$\triangle APQ \cong \triangle BPR \text{ (A.S.A.)}$$

and

$$\triangle APQ \sim \triangle BPR \text{ (trivial) Q.E.D. (ii).}$$

The figure that appeared as a part of the item (see fig. 2) is a particular case. It represents symmetry between  $A$  and  $B$  with respect to  $\overline{QR}$ , which implies that  $\angle QPA \cong \angle RPB$ . Such symmetry, however, is not given in the problem and must not be assumed. Nevertheless, the third line in the answer, where the solution says " $m\angle QPA = m\angle RPB$ ," indicates that students counted on it. Even though students should be educated not to gather information from a figure, it is somewhat unfair to put superfluous information in a test item unless the objective is to test students' ability to cope with it. The appropriate figure (see fig. 3) is not likely to lead a student possessing the level of performance demonstrated in the mistaken solution to commit an error of the type quoted. (For another problematic aspect of this item, see case 1 under "Poor Designation.")

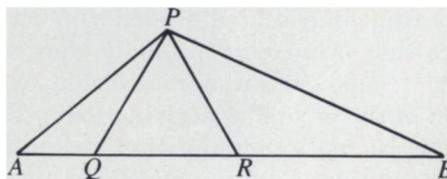


Fig. 3

Case 2

In parallelogram  $ABCD$ ,  $AB = 10$  cm,  $AD = 7$  cm.  $\overline{DC}$  is extended to  $E$ .  $CE = 6$  cm,  $BE = 5$  cm. [See fig. 4.]

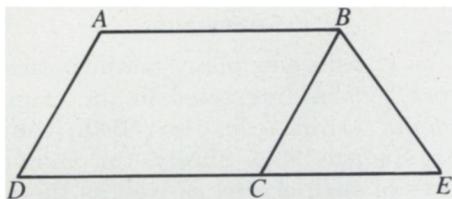


Fig. 4

- (i) Find  $m\angle BEC$ .  
 (ii) Calculate the area of the trapezoid  $ABED$ .

A repeated erroneous solution for this item was the following:

Let  $\overrightarrow{BF}$  bisect  $\angle CBE$ .  $\Rightarrow CF = FE = 3$  cm, etc.

Because triangle  $EBC$  appears in the figure as isosceles (if not equilateral), students tended to ignore temporarily the information explicitly given in the problem and relied on the figure too heavily. Later in the solution, using the data  $BE = 5$  cm, students did not notice the inconsistency. Even though the students' responsibility is to check the details carefully, misleading information in figures should be omitted. A misleading figure may be worse than no figure at all.

#### Poor Designation

##### Case 1

Triangle  $PQR$  is equilateral.  $A, B$  are two points on the continuation of  $RQ$ .  $m\angle APB = 120^\circ$ . [See fig. 2.]

- Prove: (i)  $\angle APQ \cong \angle PBR$   
 (ii)  $\triangle APQ \sim \triangle PBR$

Many students mixed the order of  $P$  and  $B$  when they referred to angles and to triangles. For instance, they wrote  $\angle BPR$  where  $\angle PBR$  was treated. We speculated that the resemblance of  $P$  to  $B$  in sound caused much of this mix-up, particularly for students who whisper to themselves or use inner speech. We found a similar phenomenon with  $M$  and  $N$  and with  $Q$  and  $U$ . When  $P$  was replaced by  $S$ , this kind of error was diminished. So simple, yet so hard to predict....

##### Case 2

Eliminate  $n$  in the formula

$$i = \frac{nE}{nR + r}$$

Quite a few students wrote this:

$$i = \frac{nE}{R(n + 1)}$$

$$\frac{n + 1}{n} = \frac{E}{iR}$$

$$1 + \frac{1}{n} = \frac{E}{iR}, \dots$$

Here  $R$  and  $r$  were considered identical. (This error, however, may be particularly typical of students whose native language does not have lowercase letters.) No such error occurred when  $R$  and  $r$  were replaced by  $R$  and  $T$ , respectively.

#### Ambiguous Phrasing

A pool of capacity  $N$  cubic meters is filled by two pipes. During  $C$  hours  $V$  cubic meters of water can enter the pool through one pipe, and  $B$  cubic meters can enter through the other pipe. How many cubic meters of water can enter the pool in one minute through both pipes?

Here is an excerpt from a student's workbook:

Both pipes are open, therefore the amount of water through each multiplied by the time ( $C$ ) equals the capacity  $N$ ; therefore:  $N = (V + B) \cdot C$ . In one minute we get

$$\frac{(V + B) \cdot C}{C} \times \frac{1}{60} = \frac{V + B}{60}$$

This answer suggests that the student read the problem carelessly, making a slight change in punctuation, as follows: "A pool of capacity  $N$  cubic meters is connected to two pipes during  $C$  hours;  $V$  cubic meters of water (per minute) can enter the pool through one pipe and  $B$  cubic meters (per minute) through the other pipe."

Even though the student's responsibility is not lessened by the item's poor phrasing,

the following is an alternative phrasing, less likely to invite trouble:

Two pipes are connected to a pool of capacity  $N$  cubic meters. One pipe fills  $V$  cubic meters of water in  $C$  hours. The other pipe fills  $B$  cubic meters in  $C$  hours. How many . . . ?

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### Conclusion

In the last decade, many mathematics educators became interested in the origins of students' errors (e.g., Hart 1978; Quintero 1983; Radatz 1980). Surely, the analysis of causes of such errors, as well as the evaluation of achievement, presupposes test validity. However, test validity might be jeopardized if the form of the test invites mistakes that could be avoided if editing was done more carefully. Sometimes a sophisticated ability to predict errors through an examination of subtle aspects of editing is required.

In this article we presented examples of students' mistakes in written tests that we attributed to four editorial factors: careless printing, poor designation, misleading figures, and ambiguous phrasing. We believe such analyses are a step toward following Suydam (1974) in the transition of evaluation from "why" to "how." The more such information is obtained and made known, the better predictions test designers can make.

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