

## ***Infinitely Many Different Quartic Polynomial Curves***

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The graph of every polynomial of first degree is an inclined straight line. The graph of every polynomial of second degree is a parabola. Such uniformity of shape does not hold for the graphs of all polynomials of third degree, as we can see in Figures 1–3. How many distinct shapes are there to the graphs of cubic polynomials? What about the quartics? And polynomials of higher degrees? These are the questions we address in this paper.

The methods stem from our search for a qualitative introduction of polynomials [3]. The results presented below are useful for a systematic study aimed at familiarizing students with the behavior of polynomials of low degrees.

### **Shape Preserving Transformations**

To classify the geometric shapes of curves in the plane into distinct categories, some type of transformation of the plane must be declared as shape preserving. We use those *affine* mappings of the Cartesian  $(x, y)$ -plane that result in uniform dilatations, reflections, and translations in the directions of the coordinate axes. (For a geometric presentation of affine transformations see [5].)

Any such affine transformation  $T$ , when applied to the graph of a real function  $f(x)$ , has the form

$$f(x) \rightarrow Tf(x) = g(x) = b_1 f(c_1 x + c_0) + b_0, \quad (1)$$

where  $b_0, b_1, c_0, c_1$  are constant real numbers, called the parameters of the transformation. The values  $c_1, b_1$  characterize the dilatations (stretchings/shrink-

ings) of the graph in the directions of the  $x$ - and  $y$ -axis respectively, and both are nonzero:

$$b_1 \neq 0, \quad c_1 \neq 0. \quad (2)$$

A negative value of  $b_1$  or  $c_1$  results in a reflection of the graph relative to the  $y$ - or the  $x$ -axis, respectively. The values  $c_0, b_0$  translate the graph parallel to the axes.

### Properties of AD-Transformations

Because all the deformations of the graph mentioned above are performed in the directions of the coordinate axes, we shall use the term *AD-transformation* of  $f(x)$  for transformation (1) with conditions (2), where *AD* stands for *Axial Deformations*.

Every AD-transformation is invertible:

$$f(x) = T^{-1}g(x) = \hat{b}_1 g(\hat{c}_1 x + \hat{c}_0) + \hat{b}_0,$$

where

$$\begin{aligned} \hat{b}_0 &= -b_0/b_1, & \hat{b}_1 &= 1/b_1, \\ \hat{c}_0 &= -c_0/c_1, & \hat{c}_1 &= 1/c_1. \end{aligned} \quad (3)$$

The product, i.e. composition, of any two AD-transformations is again an AD-transformation.

It follows that the group of AD-transformations defines an equivalence relation on the set of real-valued functions  $f(x)$ . We say that  $f(x)$  is *equivalent* to  $g(x)$  if  $Tf(x) = g(x)$  for some AD-transformation  $T$ . If  $Tf(x) = g(x)$  is given by (1), then the correspondence  $(x, y) \rightarrow (c_1 x + c_0, (y - b_0)/b_1)$  maps points on the graph of  $g(x)$  onto points on the graph of  $f(x)$ . AD-transformations preserve many properties of functions and their graphs. For example, suppose  $f(x)$  is equivalent to  $g(x)$ . Then

- (i)  $f(x)$  is monotone (convex or concave) on an interval if and only if  $g(x)$  is monotone (convex or concave) on the corresponding interval,
- (ii) a point on the graph of  $f(x)$  is an extreme point (a point of inflection) if and only if the corresponding point on the graph of  $g(x)$  is an extreme point (a point of inflection),
- (iii) the graph of  $f(x)$  is symmetric about a vertical line if and only if the graph of  $g(x)$  is symmetric about the corresponding vertical line, and
- (iv) the graph of  $f(x)$  is symmetric through a point if and only if the graph of  $g(x)$  is symmetric about the corresponding point.

Although it is clear from geometrical considerations that each of the properties stated above is preserved by an AD-transformation, one can give algebraic proofs, some of which are based on the derivative. If polynomials  $f(x)$  and  $g(x)$  are equivalent, it follows from (1) that  $f'(x)$  and  $g'(x)$  are equivalent:

$$f'(x) \rightarrow g'(x) = b_1 c_1 f'(c_1 x + c_0).$$

Thus, for example, if  $f'(x)$  is positive on an interval, then  $g'(x)$  is either entirely positive or entirely negative (depending on the sign of  $b_1 c_1$ ) on the corresponding interval. In this way the statement (i) is proved.

The uniformity of the graphs of all linear functions is due to the obvious fact that they are all AD-transformations of the function  $l_1(x) = x$ . The uniformity of the graphs of all quadratic polynomials is due to the fact that each quadratic polynomial can be represented as  $q(x) = a(x - x_0)^2 + y_0$ , and hence it is an AD-transformation of  $q_1(x) = x^2$ . Before we turn to cubics, we introduce the notion of a generic set.

### A Generic Set

If  $F$  is a set of functions, we have defined two functions  $f_1(x), f_2(x) \in F$  to be equivalent if  $f_1(x) = Tf_2(x)$  for some AD-transformation  $T$ . Accordingly, the original set  $F$  is split into equivalence classes. We will call a subset of  $F$  that contains exactly one representative from each equivalence class a *generic set* for  $F$ .

In particular, the singleton  $\{l_1(x) = x\}$  is a generic set for the linear polynomials and the singleton  $\{q_1(x) = x^2\}$  is a generic set for the quadratics.

As we shall see, any generic set for polynomials of third degree is finite and consists of three cubic polynomials, but there is no finite generic set for polynomials of fourth degree. This implies that there exist infinitely many different types of quartic curves!

### Cubics

Let  $P(x)$  denote a cubic, i.e., a polynomial defined for all  $x$  by

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \quad (a_3 \neq 0).$$

Any AD-transformation of  $P(x)$  with parameters  $b_0, b_1, c_0, c_1$  ( $b_1, c_1 \neq 0$ ) is a new cubic

$$\begin{aligned} C(x) &= b_1P(c_1x + c_0) + b_0 \\ &= b_1(a_0 + a_1(c_1x + c_0) + a_2(c_1x + c_0)^2 + a_3(c_1x + c_0)^3) + b_0. \end{aligned}$$

Hence,  $C(x) = A_0 + A_1x + A_2x^2 + A_3x^3$ , where

$$\begin{aligned} A_0 &= b_1(a_0 + a_1c_0 + a_2c_0^2 + a_3c_0^3) + b_0, \\ A_1 &= b_1c_1(a_1 + 2a_2c_0 + 3a_3c_0^2), \\ A_2 &= b_1c_1^2(a_2 + 3c_0a_3), \\ A_3 &= b_1c_1^3a_3. \end{aligned}$$

Clearly,  $C(x)$  can be simplified by a clever choice of the parameters. Because  $a_3 \neq 0$ , we can always get  $A_2 = 0$  by choosing  $c_0 = -a_2/(3a_3)$ . Because  $c_1 \neq 0$ , we can always get  $A_3 = 1$  by choosing  $b_1 = 1/(a_3c_1^3)$ . Finally, it is always possible to take

$$b_0 = -b_1(a_0 + a_1c_0 + a_2c_0^2 + a_3c_0^3)$$

so that  $A_0 = 0$ .

By choosing these values for  $b_0, b_1$  and  $c_0$ , every cubic polynomial can be transformed into a new one of the form

$$C(x) = x^3 + A_1x,$$

where

$$A_1 = (3a_1a_3 - a_2^2)/(3a_3^2c_1^2).$$

Now, set

$$D = a_2^2 - 3a_1a_3$$

so that

$$A_1 = -D/(3a_3^2c_1^2).$$

If  $D = 0$  then  $A_1 = 0$  and consequently

$$C(x) = x^3.$$

If  $D > 0$  then by choosing the fourth parameter  $c_1$  to be  $c_1 = \sqrt{D/(3a_3^2)}$  we get  $A_1 = -1$  and consequently

$$C(x) = x^3 - x.$$

If  $D < 0$ , then by taking  $c_1 = \sqrt{-D/(3a_3^2)}$  we get  $A_1 = 1$  and

$$C(x) = x^3 + x.$$

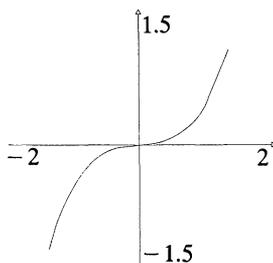
Thus, every polynomial of third degree

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

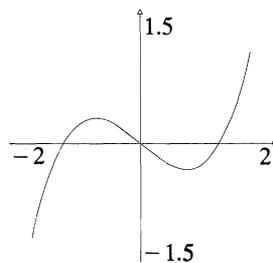
can be transformed by some AD-transformation to one of three forms:

$C_1(x) = x^3$	if $D = 0$	(see Figure 1),
$C_2(x) = x^3 - x$	if $D > 0$	(see Figure 2),
$C_3(x) = x^3 + x$	if $D < 0$	(see Figure 3).

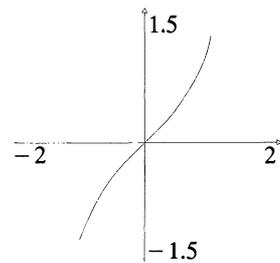
Because an AD-transformation is invertible, it follows that each cubic polynomial can be generated by an AD-transformation of one of the three cubics above. At the same time, no two of the three cubics  $C_1, C_2, C_3$  can be AD-transformed into one another, for each one possesses some special property preserved under AD-transformation which the other two do not possess:  $C_1(x)$  is increasing and its graph has one horizontal tangent (Figure 1),  $C_2(x)$  possesses two extrema (Figure 2), and  $C_3(x)$  is increasing (as is  $C_1(x)$ ) but its graph does not have a horizontal tangent (Figure 3).



**Figure 1**  
 $C_1(x) = x^3$



**Figure 2**  
 $C_2(x) = x^3 - x$

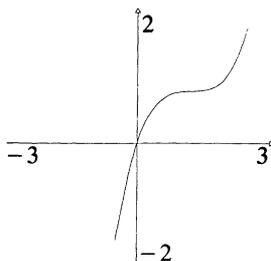


**Figure 3**  
 $C_3(x) = x^3 + x$

The cubics  $C_1, C_2, C_3$  constitute a finite generic set for polynomials of third degree. Hence there are three distinct shapes of graphs for the cubics. The shape of a given cubic is defined by the sign of the expression

$$D = a_2^2 - 3a_1a_3,$$

which thus serves as a discriminant for the shapes of the cubics. For instance, for the cubic  $P(x) = x^3 - 3x^2 + 3x$  the value of the discriminant is  $D = (-3)^2 - 3 \cdot 3 \cdot 1 = 0$ . This implies that the given cubic is equivalent to the generic cubic  $C_1(x) = x^3$  and therefore the graph of  $P(x)$  is an increasing curve with one horizontal tangent. The graph in Figure 4 confirms these predictions.



**Figure 4**

The cubic  $C(x) = x^3 - 3x^2 + 3x$  possessing a monotonic graph with one horizontal tangent

Finally, note that all generic cubics are odd functions, so their graphs are symmetric about the origin, which is their inflection point. Because AD-transformations preserve inflection points and central symmetry, it follows that each cubic

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

is symmetric about its inflection point

$$x = -a_2/(3a_3).$$

This conclusion implies that in any non-monotonic graph of a cubic polynomial, the two extremal points and the inflection point are colinear.

## Quartics

How large is a generic set for quartics? For any given quartic polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4, \text{ where } a_4 \neq 0,$$

one may replace  $x$  by  $x - a_3/(4a_4)$  to eliminate the cubic term:

$$P(x - a_3/(4a_4)) = P(-a_3/(4a_4)) + D_1x/(8a_4^2) - D_2x^2/(8a_4) + a_4x^4 \quad (4)$$

where

$$D_1 = a_3^3 - 4a_2a_3a_4 + 8a_1a_4^2,$$

$$D_2 = 3a_3^2 - 8a_2a_4.$$

If in (4) we multiply  $x$  by any nonzero real  $c_1$  and divide both sides by  $a_4c_1^4$  we get

$$b_1P(c_1x + c_0) + b_0 = x^4 - D_2x^2/(8a_4^2c_1^2) + D_1x/(8a_4^3c_1^3),$$

where

$$c_0 = -a_3/(4a_4), b_1 = 1/(a_4c_1^4), b_0 = -b_1P(c_0).$$

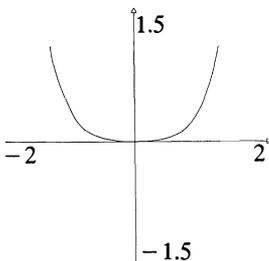
This is, in fact, an AD-transformation of the given quartic  $P(x)$  which depends on a yet undetermined parameter  $c_1$ . The values of  $D_1$  and  $D_2$  are determined by the coefficients of the original quartic. If  $D_1 = 0, D_2 = 0$  then  $Q(x) = Q_1(x) = x^4$ . If  $D_1 = 0$  and  $D_2 > 0$ , the choice of the value  $c_1 = \sqrt{D_2/(8a_4^2)}$  gives  $Q(x) = Q_2(x) = x^4 - x^2$ . If  $D_1 = 0$  and  $D_2 < 0$ , it is possible to choose  $c_1 = \sqrt{-D_2/(8a_4^2)}$  and to get  $Q(x) = Q_3(x) = x^4 + x^2$ . Finally, if  $D_1 \neq 0$  and  $c_1 = \sqrt[3]{D_1}/(2a_4)$  then  $Q(x) = Q_4(x) = x^4 + x + rx^2$ , where  $r = -D_2/(2\sqrt[3]{D_1^2})$ .

It, therefore, follows that each polynomial of fourth degree with real coefficients is equivalent to one of the following quartics:

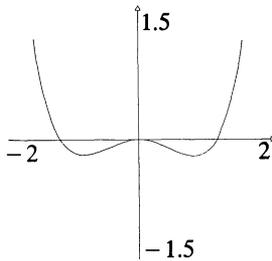
$$\begin{aligned} Q_1(x) &= x^4, \\ Q_2(x) &= x^4 - x^2, \\ Q_3(x) &= x^4 + x^2, \\ Q_4(x) &= x^4 + x + rx^2, \end{aligned} \tag{5}$$

where  $r \in R$  is a real parameter.

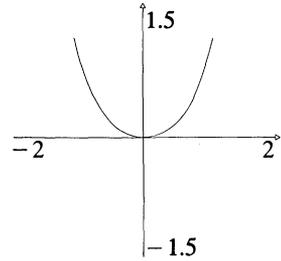
Figures 5, 6 and 7 show the graphs of the quartics  $Q_1, Q_2$  and  $Q_3$ .



**Figure 5**  
 $Q_1(x) = x^4$



**Figure 6**  
 $Q_2(x) = x^4 - x^2$



**Figure 7**  
 $Q_3(x) = x^4 + x^2$

The formula for  $Q_4$  in (5) describes an infinite one-parameter family of quartics (Figures 8a, 8b).

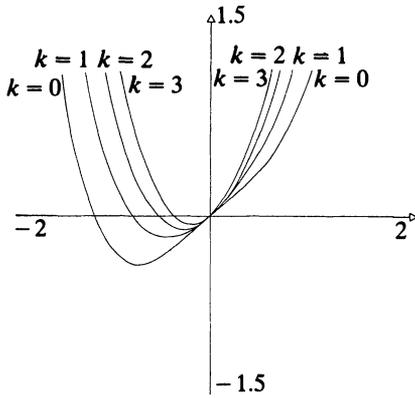
We shall prove now that no two of the quartics in (5) are equivalent to one another.

First, let us establish the mutual inequivalence of any two quartics  $Q_4(x)$  with different values of the parameter  $r$ . Suppose  $Q_4(x) = x^4 + x + rx^2$  is equivalent to  $\hat{Q}_4(x) = x^4 + x + \hat{r}x^2$ , where  $\hat{r} \neq r$ . Then there exist  $b_0, c_0, b_1, c_1$  such that

$$\hat{Q}_4(x) = b_1[(c_1x + c_0)^4 + (c_1x + c_0) + r(c_1x + c_0)^2] + b_0.$$

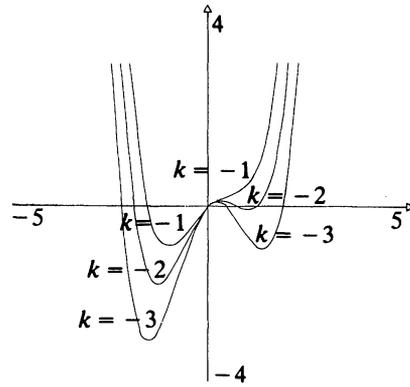
Because  $\hat{Q}_4(x)$  does not contain an  $x^3$  term,  $c_0 = 0$  and so

$$\hat{Q}_4(x) = (b_1c_1^4)x^4 + (b_1c_1)x + (b_1rc_1^2)x^2 + b_0.$$



**Figure 8a**

The one-parameter family of generic quartics  $Q_4(x) = x^4 + x + kx^2$  for  $k = 0, 1, 2, 3, \dots$



**Figure 8b**

The one-parameter family of generic quartics  $Q_4(x) = x^4 + x + kx^2$  for  $k = -1, -2, -3, \dots$

Hence,

$$b_1c_1^4 = 1, \quad b_1c_1 = 1, \quad b_1rc_1^2 = \hat{r}, \quad b_0 = 0,$$

which implies  $b_1 = c_1 = 1$ ,  $r = \hat{r}$ . Hence,  $\hat{Q}_4(x)$  must be identical to  $Q_4(x)$ . This completes the proof that no two quartics of the form  $Q_4(x)$  with different real values of  $r$  are equivalent. This phenomenon is not visually apparent at all (see Figures 8a, 8b).

In a similar manner, the fact that none of the three quartics  $Q_1(x), Q_2(x), Q_3(x)$  is equivalent to  $Q_4(x)$  can be established. The mutual inequivalence of the quartics  $Q_1, Q_2$  and  $Q_3$  can be checked by considering their derivatives  $Q_1'(x) = 4x^3$ ,  $Q_2'(x) = 4x^3 - 2x$ ,  $Q_3'(x) = 4x^3 + 2x$ . The respective values of the discriminant  $D = a_2^2 - 3a_1a_3$  of these cubics are: 0, 24 and  $-24$ . Such cubics, as we have shown, are inequivalent. Hence, so are the original quartics  $Q_1(x), Q_2(x), Q_3(x)$ . Thus we have established that *the three single quartics  $Q_1(x), Q_2(x), Q_3(x)$  together with the infinite one-parameter set of quartics  $Q_4(x)$  compose a generic set for polynomials of fourth degree.*

Because all generic sets for a set of functions must have the same cardinality, we conclude that *no generic set for polynomials of fourth degree is finite.*

Observing the graphs in Figures 8a, 8b, we might feel that we really do not want to regard them as essentially different graphs, although our equivalence relation forces us to. That is, the graphs in Figure 8a, where  $k > 0$ , look alike and the graphs in Figure 8b, where  $k < 0$ , look alike. An interesting student research project may arise here [4]. For example, is there a natural group of transformations of the plane that divides the quartics into five equivalence classes with generic set  $Q_1, Q_2, Q_3, Q_4 = x^4 + x + x^2, Q_5 = x^4 + x - x^2$ ?

## Polynomials of Higher Degrees

For any given degree  $n \geq 4$  it is impossible to transform any two polynomials of the form

$$P(x) = x^n + x + rx^2, \quad (6)$$

with distinct values of a parameter  $r$ , into one another by an AD-transformation.

The demonstration of this statement is quite similar to the one given for the case  $n = 4$ . Thus, *for polynomials of any given degree  $n > 3$  a finite generic set does not exist.*

To find an infinite generic set for polynomials of any degree  $n > 4$ , let us apply to a polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad (a_n \neq 0)$$

an AD-transformation with parameters

$$c_0 = -a_{n-1}/(na_n), \quad b_1 = 1/(a_n c_1^n), \quad b_0 = -b_1 P(c_0),$$

where  $c_1$  is yet indefinite. This transformation eliminates the term with  $x^{n-1}$  and the constant term and turns the coefficient of  $x^n$  to 1, so that

$$\begin{aligned} \hat{P}(x) &= TP(x) = b_1 P(c_1x + c_0) + b_0 \\ &= x^n + A_2x^{n-2}/c_1^2 + A_3x^{n-3}/c_1^3 + A_4x^{n-4}/c_1^4 + \cdots + A_{n-1}x/c_1^{n-1}, \end{aligned} \quad (7)$$

or

$$\begin{aligned} \hat{P}(x) &= x^n + (A_2x^{n-2}/c_1^2 + A_4x^{n-4}/c_1^4 + \cdots) \\ &\quad + (A_3x^{n-3}/c_1^3 + A_5x^{n-5}/c_1^5 + \cdots), \end{aligned} \quad (8)$$

where  $A_k$  ( $k = 2, 3, \dots, n-1$ ) are functions of  $a_1, a_2, \dots, a_n$ . Note that the degrees of all monomials in the brackets above are less than  $(n-1)$ . The degrees in the first brackets are of the same parity as  $n$ , whereas the degrees in the second brackets have the opposite parity.

Let us examine the three possible cases:

1) If  $A_k = 0$  for  $k = 2, 3, \dots, n-1$ , then it follows from (8) that

$$\hat{P}(x) = x^n. \quad (9)$$

2) If all  $A_k$  with odd indices  $k$  in the second brackets in (8) are zero, whereas among  $A_k$  with even  $k$  in the first brackets there is at least one nonzero value, let us denote by  $k = 2m$  the least of the even indices for which  $A_k \neq 0$ . Then it follows from (8) that

$$\hat{P}(x) = x^n + A_{2m}x^{n-2m}/c_1^{2m} + A_{2m+2}x^{n-2m-2}/c_1^{2m+2} + \dots$$

Choosing

$$c_1 = \sqrt[2m]{A_{2m}}, \text{ if } A_{2m} > 0$$

or

$$c_1 = \sqrt[2m]{-A_{2m}}, \text{ if } A_{2m} < 0,$$

we get from the expansion above, respectively, that

$$P(x) = x^n + x^{n-2m} + \hat{a}_{n-2m-2}x^{n-2m-2} + \hat{a}_{n-2m-4}x^{n-2m-4} + \dots, \quad (10)$$

or

$$P(x) = x^n - x^{n-2m} + \hat{a}_{n-2m-2}x^{n-2m-2} + \hat{a}_{n-2m-4}x^{n-2m-4} + \dots, \quad (11)$$

where the real coefficients  $\hat{a}_k$  are functions of the coefficients of the original polynomial  $P(x)$ .

3) If among the coefficients  $A_k$  with odd indices  $k$  in the second brackets of the expansion (8) there is at least one nonzero value, let us denote by  $k = 2m + 1$  the least of those indices for which  $A_k \neq 0$ . If we choose now

$$c_1 = \sqrt[2m+1]{A_{2m+1}},$$

the expansion (8) yields:

$$\begin{aligned} P(x) = & x^n + x^{n-2m-1} + (\hat{a}_{n-2}x^{n-2} + \hat{a}_{n-4}x^{n-4} + \dots) \\ & + (\hat{a}_{n-2m-3}x^{n-2m-3} + \hat{a}_{n-2m-5}x^{n-2m-5} + \dots). \end{aligned} \quad (12)$$

The first brackets in (12) contain all descending degrees, possessing the same parity as  $n$ , beginning with  $(n - 2)$ . The second brackets contain all descending degrees, having the opposite parity, beginning with  $(n - 2m - 3)$ .

Thus, we come to the conclusion that *every polynomial of degree  $n$  is equivalent to one of the forms (9)–(12)*.

Note that, when all polynomials of a given degree  $n$  are under consideration, the letter  $m$  in each of the expansions (10)–(12) stands for an independent parameter, which can take on any positive integer value with one limitation: all the exponents of  $x$  in the expansion must also be positive integers. For instance, if  $n = 5$ , we may take in (10) and (11)  $m = 1$  and  $m = 2$ , whereas in (12) only  $m = 1$ . For these values of  $n$  and  $m$  we conclude from (9)–(12) that each polynomial of fifth degree is equivalent to one of the following forms:

$$\hat{P}(x) = x^5,$$

$$\hat{P}(x) = x^5 + x^3 + rx,$$

$$\hat{P}(x) = x^5 - x^3 + rx,$$

$$\hat{P}(x) = x^5 + x,$$

$$\hat{P}(x) = x^5 - x,$$

$$\hat{P}(x) = x^5 + x^2 + rx + sx^3,$$

where  $r, s$  are real parameters, replacing arbitrary coefficients  $\hat{a}_1, \hat{a}_3$ .

It is possible to demonstrate, by a method similar to the one exploited earlier in the case of the quartics, that no two different polynomials described by the formulas above can be AD-transformed into one another. Thus, these formulas describe the *generic set for polynomials of fifth degree*.

In the same manner we can derive from the formulas (9)–(12) *the generic set for polynomials of sixth degree*:

$$\begin{aligned}\hat{P}(x) &= x^6, \\ \hat{P}(x) &= x^6 + x^4 + rx^2, \\ \hat{P}(x) &= x^6 - x^4 + rx^2, \\ \hat{P}(x) &= x^6 + x^2, \\ \hat{P}(x) &= x^6 - x^2, \\ \hat{P}(x) &= x^6 + x^3 + rx + sx^2 + tx^4, \\ \hat{P}(x) &= x^6 + x + rx^2 + sx^4,\end{aligned}$$

where  $r, s, t$  are real parameters.

Similarly, a generic set for polynomials of any degree  $n > 6$  can be constructed.

**Concluding Remarks.** Blinn [1], [2] studied the shapes of algebraic cubic curves and gave a classification, based upon *projective* transformations. Our study differs in two ways: on the one hand our discussion of cubic curves is limited to polynomial functions; on the other hand, we do not limit our discussion to cubics. This is because our goals and the methods they dictate are different. We look for ways to broaden and to make more effective the introduction to the study of polynomials in mathematics education. We hope that using generic sets for the study of polynomials of lower degrees is helpful in making this introduction more systematic and accessible to the students.

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### Pólya Award Winners

The recipients of the George Pólya Award for mathematical exposition in the 1990 *College Mathematical Journal* were announced at last summer's meeting of the MAA:

William B. Gearhart and Harris S. Shultz  
The Function  $\sin x/x$ , *CMJ* 21 (1990) 90–99

Mark F. Schilling  
The Longest Run of Heads, *CMJ* 21 (1990) 196–207.