

GROWTH IN GEOMETRIC JOINT ROUTINES DURING MIDDLE-SCHOOL PEER INTERACTION

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In this paper, we draw on the commognitive framework to explore types of mathematical growth during middle-school geometry peer interaction. Comparing students' routines when working apart with their joint routines when working together, we identified four types of mathematical growth. Three types were object-level growth: applicability, refinement, and flexibility. One type was a meta-level growth consisting of a shift from a configural/visual procedure to a deductive one. Our study pinpoints the types of mathematical learning that can be achieved during peer interaction and shows the ways in which they can occur. Specifically, the study shows how different types of growth can be achieved by students building on their partner's procedure in different ways.

RATIONALE

Learning through peer interaction has come to be highly regarded not only as an important 21st century skill, but also as a means to improve learning (Kuhn 2015). Studies have shown that under certain interactional conditions, such as readiness of peers to listen to each other, problem-solving in pairs or small groups can be more conducive to students' learning than solving a problem alone (e.g., Schwarz and Linchevski 2007). Other studies have examined the types of learning that can occur in peer interactions. Phelps and Damon (1989), for example, have found that peer interactions are more effective for conceptual learning and reasoning than for rote kinds of learning. Pai and colleagues (2015) showed, through the examination of pre/post-tests, that peer interaction is conducive to learners' ability to apply or adapt prior knowledge to a novel situation. Although we learn from these studies about learning in peer interaction, we still know very little about the processes of mathematical learning that take place in these interactions and about *how* these different types of learning occur. In this study, our goal is to better understand how peer interaction promotes different types of growth in students' mathematical procedures used to solve a certain problem.

THEORETICAL FRAMEWROK

The theoretical framework which we use to pursue our goal is commognition (Sfard 2008). Commognition is a sociocultural discursive framework which has been productive in studying processes of peer interactions (Chan and Sfard

2020; Sfard and Kieran 2001) as well as processes of mathematical learning (Lavie and Sfard 2019; e.g. Lavie, Steiner, and Sfard 2019). The commognitive framework conceptualizes learning as a process of routinization of students' actions (Lavie et al. 2019). Routines - repetitive patterns of actions – are thus the commognitive basic unit for analyzing learning. A routine is a task-procedure pair; it is defined as “the task, as seen by the performer, together with the procedure she executed to perform the task” (Lavie et al. 2019:161).

By studying mathematical routines, commognitive studies have been able to track learning over time and identify different types of growth in learners' routines (Lavie and Sfard 2019; Lavie et al. 2019). Flexibility is one such type of growth. A routine grows in its flexibility when another procedure is used in response to the same task. For example, Lavie and Sfard (2019) showed a growth in a young child's routine for the task "where is there more?" when in addition to the initial procedure of visually estimating two piles of cubes, the child used another procedure of aligning these cubes. The child's routine thus grew in flexibility to offer two alternative procedures for accomplishing the task. Applicability is another type of growth. Growth in applicability is detected when after applying a certain procedure to a certain task, a learner applies the same procedure to a new unfamiliar task.

Much of the growth in children's mathematical routines happens at the object level. As they become familiar with certain procedures (e.g., adding, dividing) and certain objects (e.g., natural numbers), learners gradually apply the familiar procedures to different tasks, producing an increasing number of narratives about these objects. This growth constitutes object-level learning. Yet from time to time, as students gradually get introduced to more sophisticated mathematical discourses, a meta-level change is needed (Sfard 2007). Such a meta-level change can happen when rules for substantiating mathematical narratives change, or when new objects are introduced. For example, when students get introduced to rational numbers, the familiar arithmetic rules that had so far been successfully applied to natural numbers no longer apply.

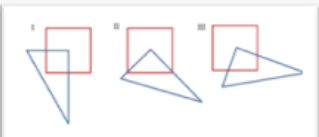
In this study, we wish to examine processes of peer interaction in junctures that afford object-level as well as meta-level learning. We pursue this goal by focusing on middle-school geometry, since a particularly critical transition is required from students in those years – the meta-level shift to deductive geometric procedures (Duval 1998). In this transition, students who are used to performing visual-configural procedures for substantiating claims about geometric objects (such as showing congruence by placing one triangle on top of the other) are required to shift to using new deductive procedures based on given data and geometric theorems (such as congruence theorems).

For examining mathematical learning in peer interaction, we add to our commognitive conceptual toolset the concept of a *joint routine* which we define

as the collection of procedures used by a group (or pair) of people working together on the same task. Based on this theoretical framework, we ask: in what ways did students' geometric joint routines grow during middle-school geometric peer interaction?

METHODOLOGY

The participants of our study were 10 middle-school students, six 8th graders (13-year-old) and four 9th graders (14-year-old), who took part in a one-hour geometric activity facilitated by the first author. The design of the activity was based on videotaped lessons of the VIDEO-LM project (Karsenty and Arcavi 2017) in which a geometric problem called *The three squares* was presented. The students in these lessons were asked to compare areas in three drawings. The canonical (correct) answer is that all areas are equal. Our design included: (1) a presentation of the geometric problem; (2) an individual session in which students worked on a worksheet (see Figure 1); (3) a dyadic session in which they worked on the same worksheet. Colored, half-transparent plastic shapes of a square and a triangle were given to the students as supporting tangible mediators.



1. In each drawing, colour the shared area of the square and triangle
2. Line up the drawings according to the size of their shared area (from the smallest to the largest)
3. Circle the correct answer in each of the following claims and explain it:
 - a. The shared area in drawing I is (smaller than /larger than/equal to/impossible to know) the shared area in drawing II
 - b. The shared area in drawing II is (smaller than /larger than/equal to/impossible to know) the shared area in drawing III
 - c. The shared area in drawing III is (smaller than /larger than/equal to/impossible to know) the shared area in drawing I

Figure 1: The worksheet

Data collected included students' 10 individual worksheets and 5 dyadic worksheets as well as footage from different cameras of both individual and dyadic sessions. Individual and dyadic sessions were fully transcribed (including non-verbal communication) and analyzed using footage from different cameras. Overall, 1530 transcription lines of verbal and non-verbal communication were analyzed.

Data analysis included the following steps: (a) analyzing students' visual mediators by adding to each line in the transcript a graphic representation of what they did, looked at and pointed to in the worksheet; (b) identifying students' procedures for the task of comparing areas when working alone, by examining students' written answers in individual worksheet as well as the

footage from their individual session and their communication at the beginning of dyadic session (c) tracking developments in dyads' joint routine for the task of comparing areas when working together, by analyzing their communication during dyadic session as well as their dyadic worksheet; (d) deductively and inductively identifying types of joint routine growth.

FINDINGS

Individually, the ten students used four main procedures for the task of comparing areas. These were: (1) the "Supplement procedure" – cutting and "moving" parts in order to supplement similar-looking shared areas; (2) the "Ratio procedure" –visually estimating the ratio of the shared area from the whole square; (3) the "Formula procedure" – visually estimating the relation between heights and bases of the shared areas and then applying to it an area formula (such as $\text{base} \times \text{height} / 2$); and (4) the "Given procedure" – examining the givens (or lack thereof) to assess if enough information is provided.

During the start of the dyadic session, the students within each dyad (dyad 1 to 5) compared their solutions with the solutions of their dyadic partners and tried to reach an agreement. Some of them used different procedures in their individual routines for comparing the shared areas.

Examining students' joint routines during dyadic session, we found four ways in which growth in these routines occurred. Three of these ways were object-level. In other words, the growth did not include a change in meta-rules. These categories of growth were: (1) applicability; (2) refinement; and (3) flexibility. Two of these growth patterns – applicability and flexibility – have been known from previous studies (Lavie and Sfard 2019; Lavie et al. 2019). Refinement is a new bottom-up category that we used to describe growth which included the refinement of specific steps in a procedure previously used by one of the students. The fourth type of growth was a meta-level shift to deductive procedures. Table 1 presents these types of growth, their description, and examples.

#	Type	Description	Example
1	Applicability (Object-level growth)	Extending application of an initial procedure to another task	The Supplement procedure, initially applied by one of the students only in relation to the comparison between shared areas I and II, was applied in dyadic sessions also to the comparison between shared areas I and III.
2	Refinement (Object-level growth)	Refining steps of an initial procedure	In the Ratio procedure, the step of visually estimating the ratio of the shared area from the whole square was refined into two separate steps: (a)

		visually estimating how many times the shared areas can fit into the square; (b) deducing the ratio of the shared area from the whole square.
3 Flexibility (Object-level growth)	Forming a new procedure based on an initial procedure (same meta-rules)	A new rotational procedure was formed based on the Ratio procedure. Both procedures, the original and the newly developed, relied on the same meta-rule of visual estimation.
4 Meta-level growth	Forming a new procedure based on an initial procedure (more developed meta-rules)	A new deductive congruence procedure was formed based on the Supplement procedure. The newly formed procedure relied on a more developed meta-rule (visual estimation is insufficient; justifications should be based on theorems and givens).

Table 1: Types of growth in joint routines during dyadic session

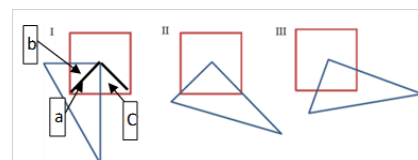
In what follows, we illustrate two of these types of growth – applicability and meta-level. We do so by focusing on the development of the most commonly used procedure – the Supplement procedure – through the case of dyad 1 (8th graders Noa and Eyal) and dyad 4 (9th graders Tamara and Orna).

Example of growth in applicability during dyad 1's session

Analyzing Noa and Eyal's initial processes in individual session, we found that Eyal only used the Ratio procedure, while Noa only used the Supplement procedure. Noa's use of the Supplement procedure was limited to the comparison between shared area I and II. Although they used different procedures to compare between shared areas I and II, they endorsed the same narrative, namely, that shared area I and shared area II are equal. Here is how Noa explained her procedure to Eyal at the start of their dyadic session:

Legend: (implied words); [parallel speech]; right column: representations of visual mediators.

36 Noa Look, these (shared areas I and II) are definitely equal 'cause... 'cause if you cut this, say, in half... here (draws line a), so what we have here (points to triangle b) you can move here (c), so we get a triangle (like shared area II) (in the picture to the right, Noa uses the plastic shapes to demonstrate more tangibly her procedure)



37 Eyal How did you think about that??



From her written answer in her individual worksheet as well as from her explanation in this excerpt, we deduced that Noa's procedure for comparing shared areas I and II included: (1) identifying the geometric shapes of the shared areas. This is evident in her reference to shared area I as "square" and to shared area II as "triangle"; (2) cutting the shape (line (a) cuts the square) of one area (area I) into sub-shapes (two triangles); (3) moving a sub-shape (triangle b) to another place (c) in the same drawing (I) so that it *supplements* a shape (triangle) similar to the other area (area II); (4) determining the relation between the shared areas (I and II) according to a visual comparison between the newly formed area (formed triangle in drawing I) and the other area (triangle in drawing II). The same procedure, with slight variations, recurred several times in students' answers, and was named the Supplement procedure.

Eyal's reaction to Noa's Supplement procedure, communicates not only that the procedure was new to him, but that he was surprised by and appreciated Noa's "thinking" (37). Following his reaction Noa suggested that they write her explanation in their shared worksheet. Eyal then said:

47 Eyal Yes, wait a second, you can cut also here (a), see? From here (a) and then put it here (b), We get this (c)

48 Noa Why?

49 Eyal To cut this, you can...

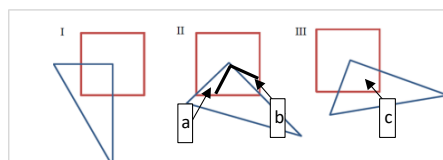
50 Noa [No but listen]

51 Eyal [take here this] small piece (a)

52 Noa [Aha]

53 Eyal [and then you] put it (a) here (b)

54 Noa But it's not enough for... [ahh right, o.k., you're right]



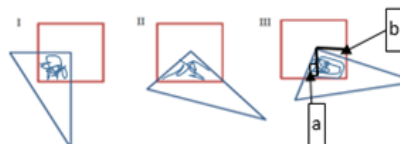
Here, Eyal applied Noa's Supplement procedure to the task of comparing between shared areas II and III. He suggested cutting the shape of one area into sub-shapes and moving a sub-shape to another place so that it supplements a similar looking shape. Therefore, Eyal did not only adopt Noa's Supplement procedure (starting his suggestion with "yes"), but also built upon it to suggest a new application (comparison between II and II) to the same procedure, an application which was not previously used by Noa. His words in line 47 communicate that he found ("wait a second") a new way of applying the same procedure ("also", "see?"). Therefore, Noa and Eyal's joint routine for comparing areas grew in applicability: from only applying the Supplement

procedure to the task of comparing areas I and II at the start of dyadic session (Noa's individual routine) to applying it also to the task of comparing between areas II and III (Noa and Eyal's joint routine).

Example of meta-level growth during dyad 4's session

Meta-level growth was found only in the interaction of Tamara and Orna, a pair of relatively high achieving 9th graders. This did not come as a complete surprise since only two interactions (Orna and Tamara's and one more dyadic interaction) were of 9th grade dyads; the other three were of 8th grade dyads, who were at the very initial stages of exposure to deductive geometric procedures. Both Tamara and Orna started out with configurally-based procedures. Orna used the Ratio procedure, while Tamara used the Supplement procedure. Toward the end of the pair's session, a meta-level growth in the dyad joint routine occurred when the girls discussed why shared area I and III are equal. The following exchange begins with Tamara suggesting using the Supplement procedure for comparing areas I and III:

132 Tamara ... you need to say that, like you move this part (a) to here (b) and then like [it will form a square]



133 Orna [I have an idea], if we, like, show congruence (between) this (a) and that (b), then... (given the context of previous utterances we interpret this as meaning: by showing that these triangles are congruent, we can show that their areas are “the same”)

...

141 Orna No, No look, you need to say that this (a) is like (meaning congruent to) this (b) in order for it to be ok to move the...

In line 132 Tamara suggested her Supplement procedure: to move part (a) so that it covers part (b) and forms a square similar to the shared area in drawing I. In response to Tamara's suggestion, Orna proposed that they use congruence theorems to substantiate that the areas of the triangles (a and b) are the same (“I have an idea, if we, like, show congruence” [133]). In line 141, Orna further explained that in order to claim that triangle (a) can be moved on top of the triangle (b) in a way that exactly covers it, they need to show that they are congruent (“you need to say... in order for...”). In other words, she did not agree (“no, no...”) with the meta-rule of the Supplement procedure that visual estimation is enough. Rather, she drew on the Supplement procedure to suggest a new deductive congruence procedure. The newly formed procedure relied on a more developed meta-rule (visual estimation is insufficient; justifications should be based on givens and theorems). By that, Tamara and Orna's joint routine for comparing areas underwent a meta-level shift.

DISCUSSION

Our goal in this study was to explore types of mathematical growth in peer interaction. Specifically, we examined developments in students' joint routines around a geometric problem that invited movement from purely configural/visual procedures to deductive ones. We found four ways in which students' routines grew during interaction. Three of these were object-level learning – applicability, refinement, and flexibility – while the fourth was a meta-level learning that included a shift from configural to deductive meta-rules. Our study contributes to commognitive research by extending the application of the study of routine growth (Lavie et al. 2019) from individuals' learning to peer learning. In addition, it adds on previous research on peer learning (Kuhn 2015) by pinpointing the types of mathematical learning that can be achieved during peer interaction, and showing the ways in which they can occur. Specifically, the study shows how different types of growth can be achieved in routines by students building on their partner's procedure in different ways.

The conclusions from this study are limited by the relatively small scope of cases, a regular limitation in studies that take such a micro-analytical look at students' discourse. Thus, future studies are needed to determine the relative frequencies of different types of joint routine growth in peer interaction. In addition, it is yet to be examined how much of what is developed jointly during students' interaction is later individualized by the participating students. Nevertheless, we believe that through our detailed theoretically anchored report, we are making progress in understanding the precise mechanisms of mathematical learning during peer interaction. A better understanding of these mechanisms of peer learning can aid educators in preparing, designing, and facilitating collaborative activities in the mathematics classroom.

References

- Chan, Man Ching Esther, and Anna Sfard. 2020. "On Learning That Could Have Happened: The Same Tale in Two Cities." *The Journal of Mathematical Behavior* 60.
- Duval, Raymond. 1998. "Geometry from a Cognitive Point of View." Pp. 37–52 in *Perspectives on the Teaching of Geometry for the 21st Century*. Vol. 5, *Perspectives on the Teaching of Geometry for the 21st Century*, edited by C. Mammana and V. Villani. Dordrecht: Springer Netherlands.
- Karsenty, Ronnie, and Abraham Arcavi. 2017. "Mathematics, Lenses and Videotapes: A Framework and a Language for Developing Reflective Practices of Teaching." *Journal of Mathematics Teacher Education* 20(5):433–55.
- Kuhn, Deanna. 2015. "Thinking Together and Alone." *Educational Researcher* 44(1):46–53.

- Lavie, Irit, and Anna Sfard. 2019. "How Children Individualize Numerical Routines: Elements of a Discursive Theory in Making." *Journal of the Learning Sciences* 28(4-5):419-61.
- Lavie, Irit, Aya Steiner, and Anna Sfard. 2019. "Routines We Live by: From Ritual to Exploration." *Educational Studies in Mathematics* 101(2):153-76.
- Pai, Hui-Hua, David A. Sears, and Yukiko Maeda. 2015. "Effects of Small-Group Learning on Transfer: A Meta-Analysis." *Educational Psychology Review*. 27(1):79-102.
- Phelps, Erin, and William Damon. 1989. "Problem Solving with Equals: Peer Collaboration as a Context for Learning Mathematics and Spatial Concepts." *Journal of Educational Psychology*. 81(4):639-46.
- Schwarz, Baruch B., and Liora Linchevski. 2007. "The Role of Task Design and Argumentation in Cognitive Development during Peer Interaction: The Case of Proportional Reasoning." *Learning and Instruction* 17(5):510-31.
- Sfard, Anna. 2008. *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*. Cambridge: Cambridge University Press.
- Sfard, Anna. 2007. "When the Rules of Discourse Change, but Nobody Tells You: Making Sense of Mathematics Learning from a Commognitive Standpoint." *Journal of the Learning Sciences* 16(4):565-613.
- Sfard, Anna, and Carolyn Kieran. 2001. "Cognition as Communication: Rethinking Learning-by-Talking through Multi-Faceted Analysis of Students' Mathematical Interactions." *Mind, Culture, and Activity* 8(1):42-76.